

On the Dirichlet problem for degenerate Beltrami equations

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ABSTRACT. We show that every homeomorphic $W_{\text{loc}}^{1,1}$ solution f to a Beltrami equation $\bar{\partial}f = \mu \partial f$ in a domain $D \subset \mathbb{C}$ is the so-called lower Q -homeomorphism with $Q(z) = K_{\mu}^T(z, z_0)$ where $K_{\mu}^T(z, z_0)$ is the tangent dilatation of f with respect to an arbitrary point $z_0 \in \bar{D}$ and develop the theory of the boundary behavior of such solutions. Then, on this basis, we show that, for wide classes of degenerate Beltrami equations $\bar{\partial}f = \mu \partial f$, there exist regular solutions of the Dirichlet problem in arbitrary Jordan domains in \mathbb{C} and pseudoregular and multi-valued solutions in arbitrary finitely connected domains in \mathbb{C} bounded by mutually disjoint Jordan curves.

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1. Introduction

Let D be a domain in the complex plane \mathbb{C} , i.e., a connected open subset of \mathbb{C} , and let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. (almost everywhere) in D . A **Beltrami equation** is an equation of the form

$$(1.1) \quad f_{\bar{z}} = \mu(z) f_z$$

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where $f_{\bar{z}} = \bar{\partial}f = (f_x + if_y)/2$, $f_z = \partial f = (f_x - if_y)/2$, $z = x + iy$, and f_x and f_y are partial derivatives of f in x and y , correspondingly. The function μ is called the **complex coefficient** and

$$(1.2) \quad K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}$$

the **dilatation quotient** of the equation (1.1). The Beltrami equation (1.1) is said to be **degenerate** if $\text{ess sup } K_\mu(z) = \infty$. The existence of homeomorphic $W_{\text{loc}}^{1,1}$ solutions was recently established to many degenerate Beltrami equations, see, e.g., related references in the recent monographs [13] and [29] and in the surveys [12] and [50].

Given a point z_0 in \mathbb{C} , we also apply here the quantity

$$(1.3) \quad K_\mu^T(z, z_0) = \frac{\left|1 - \frac{\bar{z} - \bar{z}_0}{z - z_0} \mu(z)\right|^2}{1 - |\mu(z)|^2}$$

that is called the **tangent dilatation** of the Beltrami equation (1.1) with respect to z_0 , see, e.g., [42], cf. the corresponding terms and notations in [1, 11] and [25]. Note that

$$(1.4) \quad K_\mu^{-1}(z) \leq K_\mu^T(z, z_0) \leq K_\mu(z)$$

for all $z_0 \in \mathbb{C}$ and $z \in D$.

Let us clear a geometric sense of the tangent dilatation. A point $z \in \mathbb{C}$ is called a **regular point** for a mapping $f : D \rightarrow \mathbb{C}$ if f is differentiable at z and $J_f(z) \neq 0$. Given $\omega \in \mathbb{C}$, $|\omega| = 1$, the **derivative in the direction** ω of the mapping f at the point z is

$$(1.5) \quad \partial_\omega f(z) = \lim_{t \rightarrow +0} \frac{f(z + t \cdot \omega) - f(z)}{t}.$$

The **radial direction** at a point $z \in D$ with respect to the center $z_0 \in \mathbb{C}$, $z_0 \neq z$, is

$$(1.6) \quad \omega_0 = \omega_0(z, z_0) = \frac{z - z_0}{|z - z_0|}.$$

The **tangent direction** at a point $z \in D$ with respect to the center $z_0 \in \mathbb{C}$, $z_0 \neq z$, is $\tau = i\omega_0$. The **tangent dilatation** of f at z with respect to z_0 is the quantity

$$(1.7) \quad K^T(z, z_0, f) := \frac{|\partial_T^{z_0} f(z)|^2}{|J_f(z)|},$$

where $\partial_T^{z_0} f(z)$ is the derivative of f at z in the direction τ .

Note that if z is a regular point of f and $|\mu(z)| < 1$, $\mu(z) = f_{\bar{z}}/f_z$, then

$$(1.8) \quad K^T(z, z_0, f) = K_\mu^T(z, z_0),$$

i.e.

$$(1.9) \quad K_\mu^T(z, z_0) = \frac{|\partial_T^{z_0} f(z)|^2}{|J_f(z)|}.$$

Indeed, the equalities (1.8) and (1.9) follow directly from the calculations

$$(1.10) \quad \partial_T^{z_0} f = \frac{1}{r} \left(\frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial \vartheta} + \frac{\partial f}{\partial \bar{z}} \cdot \frac{\partial \bar{z}}{\partial \vartheta} \right) = i \cdot \left(\frac{z - z_0}{|z - z_0|} \cdot f_z - \frac{\bar{z} - \bar{z}_0}{|z - z_0|} \cdot f_{\bar{z}} \right)$$

where $r = |z - z_0|$ and $\vartheta = \arg(z - z_0)$ because $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2$.

Recall that every holomorphic (analytic) function f in a domain D in \mathbb{C} satisfies the simplest Beltrami equation

$$(1.11) \quad f_{\bar{z}} = 0$$

with $\mu(z) \equiv 0$. If a holomorphic function f given in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is continuous in its closure, then by the Schwarz formula

$$(1.12) \quad f(z) = i \operatorname{Im} f(0) + \frac{1}{2\pi i} \int_{|\zeta|=1} \operatorname{Re} f(\zeta) \cdot \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta},$$

see, e.g., Section 8, Chapter III, Part 3 in [15]. Thus, the holomorphic function f in the unit disk \mathbb{D} is determined, up to a purely imaginary additive constant ic , $c = \operatorname{Im} f(0)$, by its real part $\varphi(\zeta) = \operatorname{Re} f(\zeta)$ on the boundary of \mathbb{D} .

Hence the **Dirichlet problem** for the Beltrami equation (1.1) in a domain $D \subset \mathbb{C}$ is the problem on the existence of a continuous function $f : D \rightarrow \mathbb{C}$ having partial derivatives of the first order a.e., satisfying (1.1) a.e. and such that

$$(1.13) \quad \lim_{z \rightarrow \zeta} \operatorname{Re} f(z) = \varphi(\zeta) \quad \forall \zeta \in \partial D$$

for a prescribed continuous function $\varphi : \partial D \rightarrow \mathbb{R}$, see, e.g., [2] and [53]. It is obvious that if f is a solution of this problem, then the function $F(z) = f(z) + ic$, $c \in \mathbb{R}$, is also so.

Recall that a mapping $f : D \rightarrow \mathbb{C}$ is called **discrete** if the preimage $f^{-1}(y)$ consists of isolated points for every $y \in \mathbb{C}$, and **open** if f maps every open set $U \subseteq D$ onto an open set in \mathbb{C} .

If $\varphi(\zeta) \not\equiv \text{const}$, then the **regular solution** of the Dirichlet problem (1.13) for the Beltrami equation (1.1) is a continuous, discrete and open mapping $f : D \rightarrow \mathbb{C}$ of the Sobolev class $W_{\text{loc}}^{1,1}$ with its Jacobian $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 \neq 0$ a.e. satisfying (1.1) a.e. and the condition (1.13). The regular solution of such a problem with $\varphi(\zeta) \equiv c$, $\zeta \in \partial D$, for the Beltrami equation (1.1) is the function $f(z) \equiv c$, $z \in D$.

Examples given in the paper [5] show that, even in the case of the simplest domain, the unit disk \mathbb{D} in \mathbb{C} , any power of the integrability of the dilatation K_μ does not guarantee the existence of the regular solutions of the Dirichlet problem (1.13) for the Beltrami equation (1.1) if $\varphi(\zeta) \not\equiv \text{const}$. The corresponding criteria has a much more complicated nature.

Boundary value problems for the Beltrami equations are due to the well-known Riemann dissertation in the case of $\mu(z) = 0$ and to the papers of Hilbert (1904, 1924) and Poincaré (1910) for the corresponding Cauchy–Riemann system. The Dirichlet problem was well studied for uniformly elliptic systems, see, e.g., [2] and [53]. The Dirichlet problem for degenerate Beltrami equations in the unit disk was recently studied in [5]. In comparison with this work, our approach is based on estimates of the modulus of dashed lines but not of paths under arbitrary homeomorphic $W_{\text{loc}}^{1,1}$ solutions of the Beltrami equations.

Recently in [20], it was showed that every homeomorphic $W_{\text{loc}}^{1,1}$ solution f to a Beltrami equation (1.1) in a domain $D \subset \mathbb{C}$ is the so-called lower Q -homeomorphism with $Q(z) = K_\mu(z)$ at an arbitrary point $z_0 \in \overline{D}$, and in [21], it was formulated new existence theorems for the Dirichlet problem to the Beltrami equations in terms of $K_\mu(z)$. Here we show that f is the lower Q -homeomorphism with $Q(z) = K_\mu^T(z, z_0)$ at each point $z_0 \in \overline{D}$, see further Theorem 4.1. This is

the basis for developing the theory of the boundary behavior of solutions that can be applied in turn to the research of various boundary value problems for (1.1). Namely, we prove, for wide classes of degenerate Beltrami equations (1.1), that the Dirichlet problem (1.13) has regular solutions in an arbitrary Jordan domain and pseudoregular and multi-valued solutions in an arbitrary finitely connected domain bounded by a finite collection of mutually disjoint Jordan curves. The main criteria are formulated by us in terms of the tangent dilatations $K_\mu^T(z, z_0)$ which are more refined although the corresponding criteria remain valid, in view of (1.4), for the usual dilatation $K_\mu(z)$, too.

Throughout this paper, $B(z_0, r) = \{z \in \mathbb{C} : |z_0 - z| < r\}$, $\mathbb{D} = B(0, 1)$, $S(z_0, r) = \{z \in \mathbb{C} : |z_0 - z| = r\}$, $S(r) = S(0, r)$, $R(z_0, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$.

2. Preliminaries

Recall that a real-valued function u in a domain D in \mathbb{C} is said to be of **bounded mean oscillation** in D , abbr. $u \in \text{BMO}(D)$, if $u \in L_{\text{loc}}^1(D)$ and

$$(2.1) \quad \|u\|_* := \sup_B \frac{1}{|B|} \int_B |u(z) - u_B| dm(z) < \infty,$$

where the supremum is taken over all discs B in D , $dm(z)$ corresponds to the Lebesgue measure in \mathbb{C} and

$$u_B = \frac{1}{|B|} \int_B u(z) dm(z).$$

We write $u \in \text{BMO}_{\text{loc}}(D)$ if $u \in \text{BMO}(U)$ for every relatively compact subdomain U of D (we also write BMO or BMO_{loc} if it is clear from the context what D is).

The class BMO was introduced by John and Nirenberg (1961) in the paper [19] and soon became an important concept in harmonic analysis, partial differential equations and related areas; see, e.g., [14] and [39].

A function φ in BMO is said to have **vanishing mean oscillation**, abbr. $\varphi \in \text{VMO}$, if the supremum in (2.1) taken over all balls B in D with $|B| < \varepsilon$ converges to 0 as $\varepsilon \rightarrow 0$. VMO has been introduced by Sarason in [49]. There are a number of papers devoted to the study of partial differential equations with coefficients of the class VMO , see, e.g., [4, 18, 30, 36] and [37].

REMARK 2.1. Note that $W^{1,2}(D) \subset \text{VMO}(D)$, see, e.g., [3].

Following [17], we say that a function $\varphi : D \rightarrow \mathbb{R}$ has **finite mean oscillation** at a point $z_0 \in D$, abbr. $\varphi \in \text{FMO}(z_0)$, if

$$(2.2) \quad \overline{\lim}_{\varepsilon \rightarrow 0} \int_{B(z_0, \varepsilon)} |\varphi(z) - \tilde{\varphi}_\varepsilon(z_0)| dm(z) < \infty,$$

where

$$(2.3) \quad \tilde{\varphi}_\varepsilon(z_0) = \int_{B(z_0, \varepsilon)} \varphi(z) dm(z)$$

is the mean value of the function $\varphi(z)$ over the disk $B(z_0, \varepsilon)$. Note that the condition (2.2) includes the assumption that φ is integrable in some neighborhood of the point

z_0 . We say also that a function $\varphi : D \rightarrow \mathbb{R}$ is of **finite mean oscillation in D** , abbr. $\varphi \in \text{FMO}(D)$ or simply $\varphi \in \text{FMO}$, if $\varphi \in \text{FMO}(z_0)$ for all points $z_0 \in D$. We write $\varphi \in \text{FMO}(\overline{D})$ if φ is given in a domain G in \mathbb{C} such that $\overline{D} \subset G$ and $\varphi \in \text{FMO}(G)$.

The following statement is obvious by the triangle inequality.

PROPOSITION 2.2. *If, for a collection of numbers $\varphi_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0, \varepsilon_0]$,*

$$(2.4) \quad \overline{\lim}_{\varepsilon \rightarrow 0} \int_{B(z_0, \varepsilon)} |\varphi(z) - \varphi_\varepsilon| dm(z) < \infty,$$

then φ is of finite mean oscillation at z_0 .

In particular choosing here $\varphi_\varepsilon \equiv 0$, $\varepsilon \in (0, \varepsilon_0]$, we obtain the following.

COROLLARY 2.3. *If, for a point $z_0 \in D$,*

$$(2.5) \quad \overline{\lim}_{\varepsilon \rightarrow 0} \int_{B(z_0, \varepsilon)} |\varphi(z)| dm(z) < \infty,$$

then φ has finite mean oscillation at z_0 .

Recall that a point $z_0 \in D$ is called a **Lebesgue point** of a function $\varphi : D \rightarrow \mathbb{R}$ if φ is integrable in a neighborhood of z_0 and

$$(2.6) \quad \lim_{\varepsilon \rightarrow 0} \int_{B(z_0, \varepsilon)} |\varphi(z) - \varphi(z_0)| dm(z) = 0.$$

It is known that, almost every point in D is a Lebesgue point for every function $\varphi \in L^1(D)$. Thus we have by Proposition 2.2 the following corollary.

COROLLARY 2.4. *Every locally integrable function $\varphi : D \rightarrow \mathbb{R}$ has a finite mean oscillation at almost every point in D .*

REMARK 2.5. Note that the function $\varphi(z) = \log(1/|z|)$ belongs to BMO in the unit disk Δ , see, e.g., [39], p. 5, and hence also to FMO. However, $\tilde{\varphi}_\varepsilon(0) \rightarrow \infty$ as $\varepsilon \rightarrow 0$, showing that condition (2.5) is only sufficient but not necessary for a function φ to be of finite mean oscillation at z_0 . Clearly, $\text{BMO}(D) \subset \text{BMO}_{\text{loc}}(D) \subset \text{FMO}(D)$ and as well-known $\text{BMO}_{\text{loc}} \subset L_{\text{loc}}^p$ for all $p \in [1, \infty)$, see, e.g., [19] or [39]. However, FMO is not a subclass of L_{loc}^p for any $p > 1$ but only of L_{loc}^1 . Thus, the class FMO is much more wide than BMO_{loc} .

Versions of the next lemma has been first proved for the class BMO in [41]. For the FMO case, see the papers [17, 40, 43, 44] and the monographs [12] and [29].

LEMMA 2.6. *Let D be a domain in \mathbb{C} and let $\varphi : D \rightarrow \mathbb{R}$ be a non-negative function of the class $\text{FMO}(z_0)$ for some $z_0 \in D$. Then*

$$(2.7) \quad \int_{\varepsilon < |z - z_0| < \varepsilon_0} \frac{\varphi(z) dm(z)}{\left(|z - z_0| \log \frac{1}{|z - z_0|}\right)^2} = O\left(\log \log \frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \rightarrow 0$$

for some $\varepsilon_0 \in (0, \delta_0)$ where $\delta_0 = \min(e^{-e}, d_0)$, $d_0 = \sup_{z \in D} |z - z_0|$.

Recall connections between some integral conditions, see, e.g., [46].

THEOREM 2.7. *Let $Q : \mathbb{D} \rightarrow [0, \infty]$ be a measurable function such that*

$$(2.8) \quad \int_{\mathbb{D}} \Phi(Q(z)) dm(z) < \infty$$

where $\Phi : [0, \infty] \rightarrow [0, \infty]$ is a non-decreasing convex function such that

$$(2.9) \quad \int_{\delta}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty$$

for some $\delta > \Phi(0)$. Then

$$(2.10) \quad \int_0^1 \frac{dr}{rq(r)} = \infty$$

where $q(r)$ is the average of the function $Q(z)$ over the circle $|z| = r$.

REMARK 2.8. Setting $H(t) = \log \Phi(t)$, note that by Theorem 2.1 in [46] the condition (2.9) is equivalent to each of the conditions

$$(2.11) \quad \int_{\Delta}^{\infty} H'(t) \frac{dt}{t} = \infty,$$

$$(2.12) \quad \int_{\Delta}^{\infty} \frac{dH(t)}{t} = \infty,$$

$$(2.13) \quad \int_{\Delta}^{\infty} H(t) \frac{dt}{t^2} = \infty$$

for some $\Delta > 0$,

$$(2.14) \quad \int_0^{\delta} H\left(\frac{1}{t}\right) dt = \infty$$

for some $\delta > 0$,

$$(2.15) \quad \int_{\Delta_*}^{\infty} \frac{d\eta}{H^{-1}(\eta)} = \infty$$

for some $\Delta_* > H(+0)$. Here, the integral in (2.12) is understood as the Lebesgue–Stieltjes integral and the integrals in (2.9) and (2.13)–(2.15) as the ordinary Lebesgue integrals.

The following lemma is also useful, see Lemma 2.1 in [23] or Lemma 9.2 in [29].

LEMMA 2.9. *Let (X, μ) be a measure space with a finite measure μ , $p \in (1, \infty)$ and let $\varphi : X \rightarrow (0, \infty)$ be a measurable function. Set*

$$(2.16) \quad I(\varphi, p) = \inf_{\alpha} \int_X \varphi \alpha^p d\mu$$

where the infimum is taken over all measurable functions $\alpha : X \rightarrow [0, \infty]$ such that

$$(2.17) \quad \int_X \alpha d\mu = 1.$$

Then

$$(2.18) \quad I(\varphi, p) = \left[\int_X \varphi^{-\lambda} d\mu \right]^{-\frac{1}{\lambda}}$$

where

$$(2.19) \quad \lambda = \frac{p}{p-1}, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

i.e. $\lambda = 1/(p-1) \in (0, \infty)$. Moreover, the infimum in (2.16) is attained only for the function

$$(2.20) \quad \alpha_0 = C \cdot \varphi^{-\lambda}$$

where

$$(2.21) \quad C = \left(\int_X \varphi^{-\lambda} d\mu \right)^{-1}.$$

Finally, recall that the **(conformal) modulus** of a family Γ of paths γ in \mathbb{C} is the quantity

$$(2.22) \quad M(\Gamma) = \inf_{\varrho \in \text{adm } \Gamma} \int_{\mathbb{C}} \varrho^2(z) dm(z)$$

where a Borel function $\varrho : \mathbb{C} \rightarrow [0, \infty]$ is **admissible** for Γ , write $\varrho \in \text{adm } \Gamma$, if

$$(2.23) \quad \int_{\gamma} \varrho ds \geq 1 \quad \forall \gamma \in \Gamma.$$

Here s is a natural parameter of the arc length on γ .

Later on, given sets A , B and C in \mathbb{C} , $\Delta(A, B; C)$ denotes a family of all paths $\gamma : [a, b] \rightarrow \mathbb{C}$ joining A and B in C , i.e. $\gamma(a) \in A$, $\gamma(b) \in B$ and $\gamma(t) \in C$ for all $t \in (a, b)$.

3. On regular domains

First of all, recall the following topological notion. A domain $D \subset \mathbb{C}$ is said to be **locally connected at a point** $z_0 \in \partial D$ if, for every neighborhood U of the point z_0 , there is a neighborhood $V \subseteq U$ of z_0 such that $V \cap D$ is connected. If this condition holds for all $z_0 \in \partial D$, then D is said to be **locally connected on ∂D** . Note that every Jordan domain D in \mathbb{C} is locally connected on ∂D , see, e.g., [54], p. 66.

We say that ∂D is **weakly flat at a point** $z_0 \in \partial D$ if, for every neighborhood U of the point z_0 and every number $P > 0$, there is a neighborhood $V \subset U$ of z_0 such that

$$(3.1) \quad M(\Delta(E, F; D)) \geq P$$

for all continua E and F in D intersecting ∂U and ∂V . We say that ∂D is **weakly flat** if it is weakly flat at each point $z_0 \in \partial D$.

We also say that a point $z_0 \in \partial D$ is **strongly accessible** if, for every neighborhood U of the point z_0 , there exist a compactum E in D , a neighborhood $V \subset U$ of z_0 and a number $\delta > 0$ such that

$$(3.2) \quad M(\Delta(E, F; D)) \geq \delta$$

for all continua F in D intersecting ∂U and ∂V . We say that ∂D is **strongly accessible** if each point $z_0 \in \partial D$ is strongly accessible.

Here, in the definitions of strongly accessible and weakly flat boundaries, one can take as neighborhoods U and V of a point z_0 only balls (closed or open) centered at z_0 or only neighborhoods of z_0 in another fundamental system of neighborhoods of z_0 . These conceptions can also be extended in a natural way to the case of \mathbb{C} and $z_0 = \infty$. Then we must use the corresponding neighborhoods of ∞ .

It is easy to see that if a domain D in \mathbb{C} is weakly flat at a point $z_0 \in \partial D$, then the point z_0 is strongly accessible from D . Moreover, it was proved by us that if a domain D in \mathbb{C} is weakly flat at a point $z_0 \in \partial D$, then D is locally connected at z_0 , see, e.g., Lemma 5.1 in [23] or Lemma 3.15 in [29].

The notions of strong accessibility and weak flatness at boundary points of a domain in \mathbb{C} defined in [22], see also [23] and [40], are localizations and generalizations of the corresponding notions introduced in [27] and [28], cf. with the properties P_1 and P_2 by Väisälä in [52] and also with the quasiconformal accessibility and the quasiconformal flatness by Näkki in [35]. Many theorems on a homeomorphic extension to the boundary of quasiconformal mappings and their generalizations are valid under the condition of weak flatness of boundaries. The condition of strong accessibility plays a similar role for a continuous extension of the mappings to the boundary.

A domain $D \subset \mathbb{C}$ is called a **quasiextremal distance domain**, abbr. **QED-domain**, see [9], if

$$(3.3) \quad M(\Delta(E, F; \mathbb{C})) \leq K \cdot M(\Delta(E, F; D))$$

for some $K \geq 1$ and all pairs of nonintersecting continua E and F in D .

It is well known, see, e.g., Theorem 10.12 in [52], that

$$(3.4) \quad M(\Delta(E, F; \mathbb{C})) \geq \frac{2}{\pi} \log \frac{R}{r}$$

for any sets E and F in \mathbb{C} intersecting all the circles $S(z_0, \rho)$, $\rho \in (r, R)$. Hence a QED-domain has a weakly flat boundary. One example in [29], Section 3.8, shows that the inverse conclusion is not true even in the case of simply connected domains in \mathbb{C} .

A domain $D \subset \mathbb{C}$ is called a **uniform domain** if each pair of points z_1 and $z_2 \in D$ can be joined with a rectifiable curve γ in D such that

$$(3.5) \quad s(\gamma) \leq a \cdot |z_1 - z_2|$$

and

$$(3.6) \quad \min_{i=1,2} s(\gamma(z_i, z)) \leq b \cdot \text{dist}(z, \partial D)$$

for all $z \in \gamma$ where $\gamma(z_i, z)$ is the portion of γ bounded by z_i and z , see [31]. It is known that every uniform domain is a QED-domain but there exist QED-domains that are not uniform, see [9]. Bounded convex domains and bounded domains with smooth boundaries are simple examples of uniform domains and, consequently, QED-domains as well as domains with weakly flat boundaries.

It is also often met with the so-called Lipschitz domains in the mapping theory and in the theory of differential equations. Recall first that $\varphi : U \rightarrow \mathbb{C}$ is said to be a **Lipshitz map** provided $|\varphi(z_1) - \varphi(z_2)| \leq M \cdot |z_1 - z_2|$ for some $M < \infty$ and for all z_1 and $z_2 \in U$, and a **bi-Lipshitz map** if in addition $M^* |z_1 - z_2| \leq |\varphi(z_1) - \varphi(z_2)|$ for some $M^* > 0$ and for all z_1 and $z_2 \in U$. They say that D in \mathbb{C} is a **Lipshitz domain** if every point $z_0 \in \partial D$ has a neighborhood U that can be mapped by a bi-Lipschitz homeomorphism φ onto the unit disk \mathbb{D} in \mathbb{C} such that $\varphi(\partial D \cap U)$ is the intersection of \mathbb{D} with the real axis. Note that a bi-Lipschitz homeomorphism is quasiconformal and, consequently, the modulus is quasiinvariant under such a mapping. Hence the Lipschitz domains have weakly flat boundaries.

4. On estimates of modulus of dashed lines

A continuous mapping γ of an open subset Δ of the real axis \mathbb{R} or a circle into D is called a **dashed line**, see, e.g., Section 6.3 in [29]. Note that such a set Δ consists of a countable collection of mutually disjoint intervals in \mathbb{R} . This is the motivation for the term. The notion of the modulus of a family Γ of dashed lines γ is defined similarly to (2.22). We say that a property P holds for **a.e.** (almost every) $\gamma \in \Gamma$ if a subfamily of all lines in Γ for which P fails has the modulus zero, cf. [7]. Later on, we also say that a Lebesgue measurable function $\varrho : \mathbb{C} \rightarrow [0, \infty]$ is **extensively admissible** for Γ , write $\varrho \in \text{ext adm } \Gamma$, if (2.23) holds for a.e. $\gamma \in \Gamma$, see, e.g., Section 9.2 in [29].

THEOREM 4.1. *Let f be a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1) in a domain $D \subseteq \mathbb{C}$. Then*

$$(4.1) \quad M(f\Sigma_\varepsilon) \geq \inf_{\varrho \in \text{ext adm } \Sigma_\varepsilon} \int_D \frac{\varrho^2(z)}{K_\mu^T(z, z_0)} dm(z)$$

for all $z_0 \in \overline{D}$, where $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 \in (0, d_0)$, $d_0 = \sup_{z \in D} |z - z_0|$, and Σ_ε denotes the family of dashed lines consisting of all intersections of the circles $S(z_0, r)$, $r \in (\varepsilon, \varepsilon_0)$, with D .

PROOF. Fix $z_0 \in \overline{D}$. Let B be a (Borel) set of all points z in D where f has a total differential with $J_f(z) \neq 0$. It is known that B is the union of a countable collection of Borel sets B_l , $l = 1, 2, \dots$, such that $f_l = f|_{B_l}$ is a bi-Lipschitz homeomorphism, see, e.g., Lemma 3.2.2 in [6]. With no loss of generality, we may assume that the B_l are mutually disjoint. Denote also by B_* the set of all points $z \in D$ where f has a total differential with $f'(z) = 0$.

Note that the set $B_0 = D \setminus (B \cup B_*)$ has the Lebesgue measure zero in \mathbb{C} by Gehring–Lehto–Menchoff theorem, see [8] and [34]. Hence by Theorem 2.11 in [23],

see also Lemma 9.1 in [29], $\text{length}(\gamma \cap B_0) = 0$ for a.e. paths γ in D . Let us show that $\text{length}(f(\gamma) \cap f(B_0)) = 0$ for a.e. circle γ centered at z_0 .

The latter follows from absolute continuity of f on closed subarcs of $\gamma \cap D$ for a.e. such a circle γ . Indeed, the class $W_{\text{loc}}^{1,1}$ is invariant with respect to local quasi-isometries, see, e.g., Theorem 1.1.7 in [33], and the functions in $W_{\text{loc}}^{1,1}$ is absolutely continuous on lines, see, e.g., Theorem 1.1.3 in [33]. Applying say the transformation of coordinates $\log(z - z_0)$, we come to the absolute continuity on a.e. such circle γ . Fix γ_0 on which f is absolutely continuous and $\text{length}(\gamma_0 \cap B_0) = 0$. Then $\text{length}(f(\gamma) \cap f(B_0)) = \text{length}f(\gamma_0 \cap B_0)$ and for every $\varepsilon > 0$ there is an open set ω_ε in $\gamma_0 \cap D$ such that $\gamma_0 \cap B_0 \subset \omega_\varepsilon$ with $\text{length}\omega_\varepsilon < \varepsilon$, see, e.g., Theorem III(6.6) in [48]. The open set ω_ε consists of a countable collection of open arcs γ_i of the circle γ_0 . By the construction $\sum_i \text{length}\gamma_i < \varepsilon$ and by the absolute continuity of f on γ_0 the sum $\delta = \sum_i \text{length}f(\gamma_i)$ is arbitrarily small for small enough $\varepsilon > 0$. Hence $\text{length}f(\gamma_0 \cap B_0) = 0$.

Thus, $\text{length}(\gamma_* \cap f(B_0)) = 0$ where $\gamma_* = f(\gamma)$ for a.e. circle γ centered at z_0 . Now, let $\varrho_* \in \text{adm}f(\Gamma)$ where Γ is the collection of all dashed lines $\gamma \cap D$ for such circles γ and $\varrho_* \equiv 0$ outside $f(D)$. Set $\varrho \equiv 0$ outside D and on $B_0 \cup B_*$ and

$$\varrho(z) := \varrho_*(f(z))|\partial_T^{z_0} f(z)| \quad \text{for } z \in B.$$

Arguing piecewise on B_l , we have by Theorem 3.2.5 under $m = 1$ in [6] that

$$\int_{\gamma} \varrho ds = \int_{\gamma_*} \varrho_* ds_* \geq 1 \quad \text{for a.e. } \gamma \in \Gamma$$

because $\text{length}(f(\gamma) \cap f(B_0)) = 0$ and $\text{length}(f(\gamma) \cap f(B_*)) = 0$ for a.e. $\gamma \in \Gamma$. Thus, $\varrho \in \text{ext adm } \Gamma$.

On the other hand, again arguing piecewise on B_l , we have by (1.9) that

$$\int_D \frac{\varrho^2(z)}{K_\mu^T(z, z_0)} dm(z) \leq \int_{f(D)} \varrho_*^2(w) dm(w),$$

see also Lemma III.3.3 in [26], because $\varrho(z) = 0$ on $B_0 \cup B_*$. Thus, we obtain (4.1). \square

THEOREM 4.2. *Let f be a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1) in a domain $D \subseteq \mathbb{C}$. Then*

$$(4.2) \quad M(f\Sigma_\varepsilon) \geq \int_{\varepsilon}^{\varepsilon_0} \frac{dr}{\|K_\mu^T\|_1(z_0, r)}, \quad \forall z_0 \in \overline{D}, \varepsilon \in (0, \varepsilon_0), \varepsilon_0 \in (0, d_0),$$

where $d_0 = \sup_{z \in D} |z - z_0|$, Σ_ε denotes the family of dashed lines consisting of all the intersections of the circles $S(z_0, r)$, $r \in (\varepsilon, \varepsilon_0)$, with D and

$$(4.3) \quad \|K_\mu^T\|_1(z_0, r) := \int_{D(z_0, r)} K_\mu^T(z, z_0) |dz|$$

is the norm in L_1 of $K_\mu^T(z, z_0)$ over $D(z_0, r) = \{z \in D : |z - z_0| = r\} = D \cap S(z_0, r)$.

PROOF. Indeed, for every $\varrho \in \text{ext adm } \Sigma_\varepsilon$,

$$A_\varrho(r) = \int_{D(z_0, r)} \varrho(z) |dz| \neq 0 \quad \text{a.e. in } r \in (\varepsilon, \varepsilon_0)$$

is a measurable function in the parameter r , say by the Fubini theorem. Thus, we may request the equality $A_\varrho(r) \equiv 1$ a.e. in $r \in (\varepsilon, \varepsilon_0)$ instead of (2.23) and, thus,

$$\inf_{\varrho \in \text{ext adm } \Sigma_\varepsilon} \int_{D \cap R_\varepsilon} \frac{\varrho^2(z)}{K_\mu^T(z, z_0)} dm(z) = \int_\varepsilon^{\varepsilon_0} \left(\inf_{\alpha \in I(r)} \int_{D(z_0, r)} \frac{\alpha^2(z)}{K_\mu^T(z, z_0)} |dz| \right) dr$$

where $R_\varepsilon = R(z_0, \varepsilon, \varepsilon_0)$ and $I(r)$ denotes the set of all measurable functions α on the dashed line $D(z_0, r) = S(z_0, r) \cap D$ such that

$$\int_{D(z_0, r)} \alpha(z) |dz| = 1.$$

Hence Theorem 4.2 follows by Lemma 2.9 with $X = D(z_0, r)$, the length as a measure μ on $D(z_0, r)$, $\varphi = \frac{1}{K_\mu^T}|_{D(z_0, r)}$ and $p = 2$. \square

The following lemma will be useful, too. Here we use the standard conventions $a/\infty = 0$ for $a \neq \infty$ and $a/0 = \infty$ if $a > 0$ and $a \cdot \infty = 0$, see, e.g., [48], p. 6.

LEMMA 4.3. *Under the notations of Theorem 4.2, if $\|K_\mu^T\|_1(z_0, r) \neq \infty$ for a.e. $r \in (\varepsilon, \varepsilon_0)$, then*

$$(4.4) \quad I^{-1} = \int_{A \cap D} K_\mu^T(z, z_0) \cdot \eta_0^2(|z - z_0|) dm(z) \leq \int_{A \cap D} K_\mu^T(z, z_0) \cdot \eta^2(|z - z_0|) dm(z)$$

for every measurable function $\eta : (\varepsilon, \varepsilon_0) \rightarrow [0, \infty]$ such that

$$(4.5) \quad \int_\varepsilon^{\varepsilon_0} \eta(r) dr = 1,$$

where $A = R(z_0, \varepsilon, \varepsilon_0)$ and

$$(4.6) \quad \eta_0(r) = \frac{1}{I \|K_\mu^T\|_1(z_0, r)}, \quad I = \int_\varepsilon^{\varepsilon_0} \frac{dr}{\|K_\mu^T\|_1(z_0, r)}.$$

PROOF. If $I = \infty$, then the left hand side in (4.4) is equal to zero and this inequality is obvious. Hence we may assume further that $I < \infty$. Note also that $\|K_\mu^T\|_1(z_0, r) \neq \infty$ for a.e. $r \in (\varepsilon, \varepsilon_0)$ and, consequently, $I \neq 0$. By (4.5), $\eta(r) \neq \infty$ a.e. in $(\varepsilon, \varepsilon_0)$. We have that $\eta(r) = \alpha(r)w(r)$ a.e. in $(\varepsilon, \varepsilon_0)$ where

$$\alpha(r) = \|K_\mu^T\|_1(z_0, r) \eta(r), \quad w(r) = \frac{1}{\|K_\mu^T\|_1(z_0, r)}.$$

By the Fubini theorem in the polar coordinates

$$C := \int_{A \cap D} K_\mu^T(z, z_0) \cdot \eta^2(|z - z_0|) dm(z) = \int_\varepsilon^{\varepsilon_0} \alpha^2(r) \cdot w(r) dr.$$

By Jensen's inequality with the weight $w(r)$, see, e.g., Theorem 2.6.2 in [38] applied to the convex function $\varphi(t) = t^2$ in the interval $\Omega = (r_1, r_2)$ and to the probability measure

$$\nu(E) = \frac{1}{I} \int_E w(r) dr, \quad E \subset \Omega,$$

we obtain that

$$\left(\int \alpha^2(r) w(r) dr \right)^{1/2} \geq \int \alpha(r) w(r) dr = \frac{1}{I}$$

where we have also used the fact that $\eta(r) = \alpha(r) w(r)$ satisfies (4.5). Thus, $C \geq I^{-1}$ and the proof is complete. \square

5. On a continuous extension of solutions to the boundary

THEOREM 5.1. *Let D and D' be domains in \mathbb{C} , D be bounded and locally connected on ∂D and $\partial D'$ be strongly accessible. Suppose that $f : D \rightarrow D'$ is a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1) such that*

$$(5.1) \quad \int_0^{\delta(z_0)} \frac{dr}{\|K_\mu^T\|_1(z_0, r)} = \infty \quad \forall z_0 \in \partial D$$

for some $\delta(z_0) \in (0, d(z_0))$ where $d(z_0) = \sup_{z \in D} |z - z_0|$ and

$$(5.2) \quad \|K_\mu^T\|_1(z_0, r) = \int_{D \cap S(z_0, r)} K_\mu^T(z, z_0) |dz|.$$

Then f can be extended to \overline{D} by continuity in $\overline{\mathbb{C}}$.

We assume that the function $K_\mu^T(z, z_0)$ is extended by zero outside of D in the following consequence of Theorem 5.1.

COROLLARY 5.2. *Let D and D' be domains in \mathbb{C} , D be bounded and locally connected on ∂D and $\partial D'$ be strongly accessible. Suppose that $f : D \rightarrow D'$ is a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1) such that*

$$(5.3) \quad k_{z_0}(\varepsilon) = O\left(\left[\log \frac{1}{\varepsilon} \cdot \log \log \frac{1}{\varepsilon} \cdot \dots \cdot \log \dots \log \frac{1}{\varepsilon}\right]\right) \quad \forall z_0 \in \partial D$$

as $\varepsilon \rightarrow 0$, where $k_{z_0}(\varepsilon)$ is the average of the function $K_\mu^T(z, z_0)$ over $S(z_0, \varepsilon)$. Then f can be extended to \overline{D} by continuity in $\overline{\mathbb{C}}$.

The proof of Theorem 5.1 is reduced to the following lemma.

LEMMA 5.3. *Let D and D' be domains in \mathbb{C} and let $f : D \rightarrow D'$ be a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1). Suppose that the domain D is bounded and locally connected at $z_0 \in \partial D$ and $\partial D'$ is strongly accessible at least at one point of the cluster set*

$$(5.4) \quad L := C(z_0, f) = \{w \in \overline{\mathbb{C}} : w = \lim_{k \rightarrow \infty} f(z_k), z_k \rightarrow z_0\}.$$

If the condition (5.1) holds for z_0 , then f extends to z_0 by continuity in $\overline{\mathbb{C}}$.

PROOF. Note that $L \neq \emptyset$ in view of compactness of the extended plane $\overline{\mathbb{C}}$. By the condition $\partial D'$ is strongly accessible at a point $\zeta_0 \in L$. Let us assume that there is one more point $\zeta_* \in L$ and set $U = B(\zeta_*, r_0)$ where $0 < r_0 < |\zeta_0 - \zeta_*|$.

In view of local connectedness of D at z_0 , there is a sequence of neighborhoods V_k of z_0 with domains $D_k = D \cap V_k$ and $\text{diam} V_k \rightarrow 0$ as $k \rightarrow \infty$. Choose in the domains $D'_k = fD_k$ points ζ_k and ζ_k^* with $|\zeta_0 - \zeta_k| < r_0$ and $|\zeta_0 - \zeta_k^*| > r_0$, $\zeta_k \rightarrow \zeta_0$ and $\zeta_k^* \rightarrow \zeta_*$ as $k \rightarrow \infty$. Let C_k be paths connecting ζ_k and ζ_k^* in D'_k . Note that by the construction $\partial U \cap C_k \neq \emptyset$. By the condition of the strong accessibility of the point ζ_0 from D' , there is a compactum $E \subseteq D'$ and a number $\delta > 0$ such that

$$(5.5) \quad M(\Delta(E, C_k; D')) \geq \delta$$

for large k . Without loss of generality we may assume that the last condition holds for all $k = 1, 2, \dots$. Note that $C = f^{-1}E$ is a compactum in D' and hence $\varepsilon_0 = \text{dist}(z_0, C) > 0$.

Let Γ_ε be the family of all paths connecting the circles $S_\varepsilon = \{z \in \mathbb{C} : |z - z_0| = \varepsilon\}$ and $S_0 = \{z \in \mathbb{C} : |z - z_0| = \varepsilon_0\}$ in D where $\varepsilon \in (0, \varepsilon_0)$ and $\varepsilon_0 = \delta(z_0)$. Note that $C_k \subset fB_\varepsilon$ for every fixed $\varepsilon \in (0, \varepsilon_0)$ for large k where $B_\varepsilon = B(z_0, \varepsilon)$. Thus, $M(f\Gamma_\varepsilon) \geq \delta$ for all $\varepsilon \in (0, \varepsilon_0)$. However, by [16] and [55],

$$(5.6) \quad M(f\Gamma_\varepsilon) \leq \frac{1}{M(f\Sigma_\varepsilon)}$$

where Σ_ε is the family of all dashed lines $D(r) := \{z \in D : |z - z_0| = r\}$, $r \in (\varepsilon, \varepsilon_0)$. Thus, $M(f\Gamma_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ by Theorem 4.2 in view of (5.1). The latter contradicts (5.5). This contradiction disproves the above assumption. \square

Combining Lemmas 5.3 and 4.3, we come to the following general lemma where we assume that the function $K_\mu^T(z, z_0)$ is extended by zero outside of the domain D .

LEMMA 5.4. *Let D and D' be domains in \mathbb{C} , D be locally connected on ∂D and $\partial D'$ be strongly accessible. Suppose that $f : D \rightarrow D'$ is a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1) such that $\|K_\mu^T\|_1(z_0, r) \neq \infty$ for a.e. $r \in (0, \varepsilon_0)$ and*

$$(5.7) \quad \int_{\varepsilon < |z - z_0| < \varepsilon_0} K_\mu^T(z, z_0) \cdot \psi_{z_0, \varepsilon}^2(|z - z_0|) dm(z) = o(I_{z_0}^2(\varepsilon)) \quad \text{as } \varepsilon \rightarrow 0 \quad \forall z_0 \in \partial D$$

for some $\varepsilon_0 \in (0, \delta_0)$ where $\delta_0 = \delta(z_0) = \sup_{z \in D} |z - z_0|$ and $\psi_{z_0, \varepsilon}(t)$ is a family of non-negative measurable (by Lebesgue) functions on $(0, \infty)$ such that

$$(5.8) \quad I_{z_0}(\varepsilon) := \int_{\varepsilon}^{\varepsilon_0} \psi_{z_0, \varepsilon}(t) dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Then f can be extended to \overline{D} by continuity in $\overline{\mathbb{C}}$.

Choosing in Lemma 5.4 $\psi(t) = 1/(t \log(1/t))$, we obtain by Lemma 2.6 the following result.

THEOREM 5.5. *Let D and D' be domains in \mathbb{C} , D be bounded and locally connected on ∂D and $\partial D'$ be strongly accessible. Suppose that $f : D \rightarrow D'$ is a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1) such that $K_\mu^T(z, z_0) \leq Q_{z_0}(z)$ a.e. in D for a function $Q_{z_0} : \mathbb{C} \rightarrow [0, \infty]$ in the class $\text{FMO}(z_0)$ at each point $z_0 \in \partial D$. Then f can be extended to \overline{D} by continuity in $\overline{\mathbb{C}}$.*

COROLLARY 5.6. *In particular, the conclusion of Theorem 5.5 holds if every point $z_0 \in \partial D$ is the Lebesgue point of a function $Q_{z_0} : \mathbb{C} \rightarrow [0, \infty]$ which is integrable in a neighborhood U_{z_0} of the point z_0 such that $K_\mu^T(z, z_0) \leq Q_{z_0}(z)$ a.e. in $D \cap U_{z_0}$.*

We assume that the function $K_\mu^T(z, z_0)$ is extended by zero outside of D in the following consequences of Theorem 5.5.

COROLLARY 5.7. *Let D and D' be domains in \mathbb{C} , D be bounded and locally connected on ∂D and $\partial D'$ be strongly accessible. Suppose that $f : D \rightarrow D'$ is a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1) such that*

$$(5.9) \quad \overline{\lim}_{\varepsilon \rightarrow 0} \int_{B(z_0, \varepsilon)} K_\mu^T(z, z_0) dm(z) < \infty \quad \forall z_0 \in \partial D.$$

Then f can be extended to \overline{D} by continuity in $\overline{\mathbb{C}}$.

COROLLARY 5.8. *Let D and D' be domains in \mathbb{C} , D be bounded and locally connected on ∂D and $\partial D'$ be strongly accessible. Suppose that $f : D \rightarrow D'$ is a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1) such that*

$$(5.10) \quad k_{z_0}(\varepsilon) = O\left(\log \frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \rightarrow 0 \quad \forall z_0 \in \partial D$$

where $k_{z_0}(\varepsilon)$ is the average of the function $K_\mu^T(z, z_0)$ over $S(z_0, \varepsilon)$. Then f can be extended to \overline{D} by continuity in $\overline{\mathbb{C}}$.

REMARK 5.9. In particular, the conclusion of Corollary 5.8 holds if

$$(5.11) \quad K_\mu^T(z, z_0) = O\left(\log \frac{1}{|z - z_0|}\right) \quad \text{as } z \rightarrow z_0 \quad \forall z_0 \in \partial D.$$

Similarly, choosing in Lemma 5.4 the function $\psi(t) = 1/t$, we come to the following statement.

THEOREM 5.10. *Let D and D' be domains in \mathbb{C} , D be bounded and locally connected on ∂D and $\partial D'$ be strongly accessible. Suppose that $f : D \rightarrow D'$ is a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1) such that*

$$(5.12) \quad \int_{\varepsilon < |z - z_0| < \varepsilon_0} K_\mu^T(z, z_0) \frac{dm(z)}{|z - z_0|^2} = o\left(\left[\log \frac{1}{\varepsilon}\right]^2\right) \quad \forall z_0 \in \partial D$$

as $\varepsilon \rightarrow 0$ for some $\varepsilon_0 = \delta(z_0) \in (0, d(z_0))$ where $d(z_0) = \sup_{z \in D} |z - z_0|$. Then f can be extended to \overline{D} by continuity in $\overline{\mathbb{C}}$.

REMARK 5.11. Choosing in Lemma 5.4 the function $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$, we are able to replace (8.6) by

$$(5.13) \quad \int_{\varepsilon < |z - z_0| < \varepsilon_0} \frac{K_\mu^T(z, z_0) dm(z)}{\left(|z - z_0| \log \frac{1}{|z - z_0|}\right)^2} = o\left(\left[\log \log \frac{1}{\varepsilon}\right]^2\right)$$

In general, we are able to give here the whole scale of the corresponding conditions in log using functions $\psi(t)$ of the form $1/(t \log 1/t \cdot \log \log 1/t \cdot \dots \cdot \log \dots \log 1/t)$.

Finally, combining Theorems 2.7 and 5.1, we obtain the following.

THEOREM 5.12. *Let D and D' be domains in \mathbb{C} , D be locally connected on ∂D and $\partial D'$ be strongly accessible. Suppose that $f : D \rightarrow D'$ is a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1) such that*

$$(5.14) \quad \int_{D \cap U_{z_0}} \Phi_{z_0}(K_\mu^T(z, z_0)) dm(z) < \infty \quad \forall z_0 \in \partial D$$

for a convex non-decreasing function $\Phi_{z_0} : [0, \infty] \rightarrow [0, \infty]$ and a neighborhood U_{z_0} of the point z_0 . If

$$(5.15) \quad \int_{\delta(z_0)}^{\infty} \frac{d\tau}{\tau \Phi_{z_0}^{-1}(\tau)} = \infty \quad \forall z_0 \in \partial D$$

for some $\delta(z_0) > \Phi_{z_0}(0)$. Then f can be extended to \overline{D} by continuity in $\overline{\mathbb{C}}$.

COROLLARY 5.13. *In particular, the conclusion of Theorem 5.12 holds if*

$$(5.16) \quad \int_{D \cap U_{z_0}} e^{\alpha(z_0) K_\mu^T(z, z_0)} dm(z) < \infty \quad \forall z_0 \in \partial D$$

for some $\alpha(z_0) > 0$ and a neighborhood U_{z_0} of the point z_0 .

6. The extension of the inverse mappings to the boundary

Let us start from the following fine lemma.

LEMMA 6.1. *Let D and D' be domains in \mathbb{C} , z_1 and z_2 be distinct points in ∂D , $z_1 \neq \infty$, and let $f : D \rightarrow D'$ be a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1). Suppose that the function $K_\mu^T(z, z_1)$ is integrable on the dashed lines*

$$(6.1) \quad D(z_1, r) := \{z \in D : |z - z_1| = r\} = D \cap S(z_1, r)$$

for some set E of numbers $r < |z_1 - z_2|$ of a positive linear measure. If D is locally connected at z_1 and z_2 and $\partial D'$ is weakly flat, then

$$(6.2) \quad C(z_1, f) \cap C(z_2, f) = \emptyset.$$

PROOF. Without loss of generality, we may assume that the domain D is bounded. Let $d = |z_1 - z_2|$. Choose $\varepsilon_0 \in (0, d)$ and $\varepsilon \in (0, \varepsilon_0)$ such that

$$E_0 := \{r \in E : r \in (\varepsilon, \varepsilon_0)\}$$

has a positive measure. The choice is possible because of a countable subadditivity of the linear measure and because of the exhaustion of E by the sets

$$E_m := \{r \in E : r \in (1/m, d - 1/m)\}.$$

Note that each of the circles $S(z_1, r)$, $r \in E_0$, separates the points z_1 and z_2 in \mathbb{C} and $D(r)$, $r \in E_0$, in D . Thus, by Theorem 4.2 we have that

$$(6.3) \quad M(f\Sigma_\varepsilon) > 0$$

where Σ_ε denotes the family of all intersections of D with the circles

$$S(z_1, r) = \{z \in \mathbb{C} : |z - z_1| = r\}, \quad r \in (\varepsilon, \varepsilon_0).$$

For $i = 1, 2$, let C_i be the cluster set $C(z_i, f)$ and suppose that $C_1 \cap C_2 \neq \emptyset$. Since D is locally connected at z_1 and z_2 , there exist neighborhoods U_i of z_i such that $W_i = D \cap U_i$, $i = 1, 2$ are connected and $U_1 \subset B(z_1, \varepsilon)$ and $U_2 \subset \mathbb{C} \setminus B(z_1, \varepsilon_0)$. Set $\Gamma = \Delta(\overline{W_1}, \overline{W_2}; D)$. By [16] and [55] and (6.3)

$$(6.4) \quad M(f\Gamma) \leq \frac{1}{M(f\Sigma_\varepsilon)} < \infty.$$

Let $\zeta_0 \in C_1 \cap C_2$. Without loss of generality, we may assume that $\zeta_0 \neq \infty$ because in the contrary case one can use an additional Möbius transformation. Choose $r_0 > 0$ such that $S(\zeta_0, r_0) \cap fW_1 \neq \emptyset$ and $S(\zeta_0, r_0) \cap fW_2 \neq \emptyset$.

By the condition $\partial D'$ is weakly flat and hence, given a finite number $M_0 > M(f\Gamma)$, there is $r_* \in (0, r_0)$ such that

$$M(\Delta(E, F; D')) \geq M_0$$

for all continua E and F in D' intersecting the circles $S(\zeta_0, r_0)$ and $S(\zeta_0, r_*)$. However, these circles can be connected by paths P_1 and P_2 in the domains fW_1 and fW_2 , respectively, and for those paths

$$M_0 \leq M(\Delta(P_1, P_2; D')) \leq M(f\Gamma).$$

The contradiction disproves the above assumption that $C_1 \cap C_2 \neq \emptyset$. The proof is complete. \square

As an immediate consequence of Lemma 6.1, we have the following statement.

THEOREM 6.2. *Let D and D' be domains in \mathbb{C} , D locally connected on ∂D and $\partial D'$ weakly flat. Suppose that $f : D \rightarrow D'$ is a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1) with $K_\mu^T(z, z_0) \in L^1(D \cap U_{z_0})$ for a neighborhood U_{z_0} of every point $z_0 \in \partial D$. Then f^{-1} has an extension to $\overline{D'}$ by continuity in $\overline{\mathbb{C}}$.*

PROOF. By the Fubini theorem with notations from Lemma 6.1, the set

$$(6.5) \quad E = \{r \in (0, d) : K_\mu^T(z, z_0)|_{D(z_0, r)} \in L^1(D(z_0, r))\}$$

has a positive linear measure because $K_\mu^T(z, z_0) \in L^1(D \cap U_{z_0})$. Consequently, arguing by contradiction, we obtain the desired conclusion on the basis of Lemma 6.1. \square

Moreover, by Lemma 6.1 we obtain also the following conclusion.

THEOREM 6.3. *Let D and D' be domains in \mathbb{C} , D bounded and locally connected on ∂D and $\partial D'$ weakly flat. Suppose that $f : D \rightarrow D'$ is a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1) with the coefficient μ such that the condition (5.1) holds for all $z_0 \in \partial D$. Then there is an extension of f^{-1} to $\overline{D'}$ by continuity in $\overline{\mathbb{C}}$.*

Combining Theorem 6.3 and Lemma 4.3, we come to the following general lemma where we assume as above that $K_\mu^T(z, z_0)$ is extended by zero outside of the domain D .

LEMMA 6.4. *Let D and D' be domains in \mathbb{C} , D be locally connected on ∂D and $\partial D'$ be a weakly flat. Suppose that $f : D \rightarrow D'$ is a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1) such that $\|K_\mu^T\|(z_0, r) \neq \infty$ for a.e. $r \in (0, \varepsilon_0)$ and*

$$(6.6) \quad \int_{\varepsilon < |z-z_0| < \varepsilon_0} K_\mu^T(z, z_0) \cdot \psi_{z_0, \varepsilon}^2(|z-z_0|) dm(z) = o(I_{z_0}^2(\varepsilon)) \quad \text{as } \varepsilon \rightarrow 0 \quad \forall z_0 \in \partial D$$

for some $\varepsilon_0 \in (0, \delta_0)$ where $\delta_0 = \delta(z_0) = \sup_{z \in D} |z - z_0|$ and $\psi_{z_0, \varepsilon}(t)$ is a family of non-negative measurable (by Lebesgue) functions on $(0, \infty)$ such that

$$(6.7) \quad I_{z_0}(\varepsilon) := \int_{\varepsilon}^{\varepsilon_0} \psi_{z_0, \varepsilon}(t) dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Then there is an extension of f^{-1} to $\overline{D'}$ by continuity in $\overline{\mathbb{C}}$.

7. On a homeomorphic extension to the boundary

Combining Lemmas 5.4 and 6.4, we obtain one more important general lemma where as we assume that $K_\mu^T(z, z_0)$ is extended by zero outside D .

LEMMA 7.1. *Let D and D' be domains in \mathbb{C} , D be locally connected on ∂D and $\partial D'$ be weakly flat. Suppose that $f : D \rightarrow D'$ is a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1) such that $\|K_\mu^T\|(z_0, r) \neq \infty$ for a.e. $r \in (0, \varepsilon_0)$ and*

$$(7.1) \quad \int_{\varepsilon < |z-z_0| < \varepsilon_0} K_\mu^T(z, z_0) \cdot \psi_{z_0, \varepsilon}^2(|z-z_0|) dm(z) = o(I_{z_0}^2(\varepsilon)) \quad \text{as } \varepsilon \rightarrow 0 \quad \forall z_0 \in \partial D$$

for some $\varepsilon_0 \in (0, \delta_0)$ where $\delta_0 = \delta(z_0) = \sup_{z \in D} |z - z_0|$ and $\psi_{z_0, \varepsilon}(t)$ is a family of non-negative measurable (by Lebesgue) functions on $(0, \infty)$ such that

$$(7.2) \quad I_{z_0}(\varepsilon) := \int_{\varepsilon}^{\varepsilon_0} \psi_{z_0, \varepsilon}(t) dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Then f can be extended to a homeomorphism $\overline{f} : \overline{D} \rightarrow \overline{D'}$ by continuity in $\overline{\mathbb{C}}$.

Arguing as in Section 5 and combining the corresponding results of Sections 5 and 6, we obtain the following results.

THEOREM 7.2. *Let D and D' be domains in \mathbb{C} , D be bounded and locally connected on ∂D and $\partial D'$ be weakly flat. Suppose that $f : D \rightarrow D'$ is a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1) such that $K_\mu^T(z, z_0) \leq Q_{z_0}(z)$ a.e. in D for a function $Q_{z_0} : \mathbb{C} \rightarrow [0, \infty]$ in the class $\text{FMO}(z_0)$ at each point $z_0 \in \partial D$. Then f can be extended to a homeomorphism $\bar{f} : \bar{D} \rightarrow \bar{D}'$ by continuity in $\bar{\mathbb{C}}$.*

COROLLARY 7.3. *In particular, the conclusion of Theorem 7.2 holds if every point $z_0 \in \partial D$ is the Lebesgue point of a function $Q_{z_0} : \mathbb{C} \rightarrow [0, \infty]$ which is integrable in a neighborhood U_{z_0} of the point z_0 such that $K_\mu^T(z, z_0) \leq Q_{z_0}(z)$ a.e. in $D \cap U_{z_0}$.*

COROLLARY 7.4. *Let D and D' be domains in \mathbb{C} , D be bounded and locally connected on ∂D and $\partial D'$ be weakly flat. Suppose that $f : D \rightarrow D'$ is a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1) such that*

$$(7.3) \quad \overline{\lim}_{\varepsilon \rightarrow 0} \int_{B(z_0, \varepsilon)} K_\mu^T(z, z_0) dm(z) < \infty \quad \forall z_0 \in \partial D.$$

Then f can be extended to a homeomorphism $\bar{f} : \bar{D} \rightarrow \bar{D}'$ by continuity in $\bar{\mathbb{C}}$.

COROLLARY 7.5. *Let D and D' be domains in \mathbb{C} , D be bounded and locally connected on ∂D and $\partial D'$ be weakly flat. Suppose that $f : D \rightarrow D'$ is a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1) such that*

$$(7.4) \quad k_{z_0}(\varepsilon) = O\left(\log \frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \rightarrow 0 \quad \forall z_0 \in \partial D$$

where $k_{z_0}(\varepsilon)$ is the average of the function $K_\mu^T(z, z_0)$ over $S(z_0, \varepsilon)$. Then f can be extended to a homeomorphism $\bar{f} : \bar{D} \rightarrow \bar{D}'$ by continuity in $\bar{\mathbb{C}}$.

REMARK 7.6. In particular, the conclusion of Corollary 7.5 holds if

$$(7.5) \quad K_\mu^T(z, z_0) = O\left(\log \frac{1}{|z - z_0|}\right) \quad \text{as } z \rightarrow z_0 \quad \forall z_0 \in \partial D.$$

THEOREM 7.7. *Let D and D' be domains in \mathbb{C} , D be bounded and locally connected on ∂D and $\partial D'$ be weakly flat. Suppose that $f : D \rightarrow D'$ is a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1) such that*

$$(7.6) \quad \int_{\varepsilon < |z - z_0| < \varepsilon_0} K_\mu^T(z, z_0) \frac{dm(z)}{|z - z_0|^2} = o\left(\left[\log \frac{1}{\varepsilon}\right]^2\right) \quad \forall z_0 \in \partial D$$

as $\varepsilon \rightarrow 0$ for some $\varepsilon_0 = \delta(z_0) \in (0, d(z_0))$ where $d(z_0) = \sup_{z \in D} |z - z_0|$. Then f can be extended to a homeomorphism $\bar{f} : \bar{D} \rightarrow \bar{D}'$ by continuity in $\bar{\mathbb{C}}$.

REMARK 7.8. Choosing in Lemma 7.1 the function $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$, we are able to replace (7.6) by

$$(7.7) \quad \int_{\varepsilon < |z - z_0| < \varepsilon_0} \frac{K_\mu^T(z, z_0) dm(z)}{\left(|z - z_0| \log \frac{1}{|z - z_0|}\right)^2} = o\left(\left[\log \log \frac{1}{\varepsilon}\right]^2\right)$$

Again, we are able also to give here the whole scale of the corresponding conditions in log using functions $\psi(t)$ of the form $1/(t \log 1/t \cdot \log \log 1/t \cdot \dots \cdot \log \dots \log 1/t)$.

THEOREM 7.9. *Let D and D' be domains in \mathbb{C} , D be bounded and locally connected on ∂D and $\partial D'$ be weakly flat. Suppose that $f : D \rightarrow D'$ is a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1) such that*

$$(7.8) \quad \int_0^{\delta(z_0)} \frac{dr}{\|K_\mu^T\|_1(z_0, r)} = \infty \quad \forall z_0 \in \partial D$$

for some $\delta(z_0) \in (0, d(z_0))$ where $d(z_0) = \sup_{z \in D} |z - z_0|$ and

$$(7.9) \quad \|K_\mu^T\|_1(z_0, r) = \int_{D \cap S(z_0, r)} K_\mu^T(z, z_0) |dz|.$$

Then f can be extended to a homeomorphism $\bar{f} : \bar{D} \rightarrow \bar{D}'$ by continuity in $\bar{\mathbb{C}}$.

COROLLARY 7.10. *Let D and D' be domains in \mathbb{C} , D be bounded and locally connected on ∂D and $\partial D'$ be weakly flat. Suppose that $f : D \rightarrow D'$ is a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1) such that*

$$(7.10) \quad k_{z_0}(\varepsilon) = O\left(\left[\log \frac{1}{\varepsilon} \cdot \log \log \frac{1}{\varepsilon} \cdot \dots \cdot \log \dots \log \frac{1}{\varepsilon}\right]\right) \quad \forall z_0 \in \partial D$$

as $\varepsilon \rightarrow 0$, where $k_{z_0}(\varepsilon)$ is the average of the function $K_\mu^T(z, z_0)$ over $S(z_0, \varepsilon)$. Then f can be extended to a homeomorphism $\bar{f} : \bar{D} \rightarrow \bar{D}'$ by continuity in $\bar{\mathbb{C}}$.

As usual, we assume here that $K_\mu^T(z, z_0)$ is extended by zero outside of D .

THEOREM 7.11. *Let D and D' be domains in \mathbb{C} , D be locally connected on ∂D and $\partial D'$ be weakly flat. Suppose that $f : D \rightarrow D'$ is a homeomorphic $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation (1.1) such that*

$$(7.11) \quad \int_{D \cap U_{z_0}} \Phi_{z_0}(K_\mu^T(z, z_0)) \, dm(z) < \infty \quad \forall z_0 \in \partial D$$

for a convex non-decreasing function $\Phi_{z_0} : [0, \infty] \rightarrow [0, \infty]$ and a neighborhood U_{z_0} of the point z_0 . If

$$(7.12) \quad \int_{\delta(z_0)}^\infty \frac{d\tau}{\tau \Phi_{z_0}^{-1}(\tau)} = \infty \quad \forall z_0 \in \partial D$$

for some $\delta(z_0) > \Phi_{z_0}(0)$. Then f can be extended to a homeomorphism $\bar{f} : \bar{D} \rightarrow \bar{D}'$ by continuity in $\bar{\mathbb{C}}$.

COROLLARY 7.12. *In particular, the conclusion of Theorem 7.9 holds if*

$$(7.13) \quad \int_{D \cap U_{z_0}} e^{\alpha(z_0) K_\mu^T(z, z_0)} \, dm(z) < \infty \quad \forall z_0 \in \partial D$$

for some $\alpha(z_0) > 0$ and a neighborhood U_{z_0} of the point z_0 .

The above results are far reaching generalizations of the well-known Gehring–Martio theorem on a homeomorphic extension to the closure of quasiconformal mappings between quasiextremal distance domains, see [9], see also [32], specified to the plane.

REMARK 7.13. By Theorem 5.1 and Remark 5.1 in [24] the condition (7.12) are not only sufficient but also necessary for continuous extension to the boundary of f with the integral constraint (7.11). Note, by Remark 2.8 the condition (7.12) is equivalent to each of the conditions (2.11)–(2.15) with $\Phi(z) = \Phi_{z_0}(z)$.

Finally, note that all the above results can be formulated in terms of the dilatation $K_\mu(z)$ instead of $K_\mu^T(z, z_0)$ because $K_\mu^T(z, z_0) \leq K_\mu(z)$ for all $z_0 \in \mathbb{C}$ and $z \in D$. These results are true, in particular, for many regular domains D and D' as convex, smooth and Lipschitz, uniform and quasiextremal distance domains by Gehring–Martio. It is important for further applications to the Dirichlet problem for the degenerate Beltrami equations that the first of the domains D can be an arbitrary Jordan domain or a bounded domain whose boundary consists of a finite collection of mutually disjoint Jordan curves.

8. On regular solutions for the Dirichlet problem in the Jordan domains

In this section, we prove that a regular solution of the Dirichlet problem (1.13) exists for every continuous function $\varphi : \partial D \rightarrow \mathbb{R}$ for wide classes of the degenerate Beltrami equations (1.1) in an arbitrary Jordan domain D . The main criteria are formulated by us in terms of the tangent dilatations $K_\mu^T(z, z_0)$ which are more refined although the corresponding criteria remain valid, in view of (1.4), for the usual dilatation $K_\mu(z)$, cf. [21]. We assume that the dilatations $K_\mu^T(z, z_0)$ and $K_\mu(z)$ are extended by zero outside of the domain D in the following lemma and remark.

LEMMA 8.1. *Let D be a Jordan domain in \mathbb{C} . Suppose that $\mu : D \rightarrow \mathbb{C}$ is a measurable function with $|\mu(z)| < 1$ a.e., $K_\mu(z) \in L_{\text{loc}}^1(D)$, $K_\mu^T(z, z_0)$ is integrable over $D \cap S(z_0, r)$ for a.e. $r \in (0, \varepsilon_0)$ and*

$$(8.1) \quad \int_{\varepsilon < |z - z_0| < \varepsilon_0} K_\mu^T(z, z_0) \cdot \psi_{z_0, \varepsilon}^2(|z - z_0|) dm(z) = o(I_{z_0}^2(\varepsilon)) \quad \text{as } \varepsilon \rightarrow 0 \quad \forall z_0 \in \overline{D}$$

for some $\varepsilon_0 \in (0, \delta_0)$ where $\delta_0 = \delta(z_0) = \sup_{z \in D} |z - z_0|$ and $\psi_{z_0, \varepsilon}(t)$ is a family of non-negative measurable (by Lebesgue) functions on $(0, \infty)$ such that

$$(8.2) \quad I_{z_0}(\varepsilon) := \int_{\varepsilon}^{\varepsilon_0} \psi_{z_0, \varepsilon}(t) dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Then the Beltrami equation (1.1) has a regular solution f of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$.

REMARK 8.2. Note that if the family of the functions $\psi_{z_0, \varepsilon}(t) \equiv \psi(t)$ is independent on the parameters z_0 and ε , then the condition (8.1) implies that $I_{z_0}(\varepsilon) \rightarrow \infty$

as $\varepsilon \rightarrow 0$. This follows immediately from arguments by contradiction. Note also that (8.1) holds, in particular, if

$$(8.3) \quad \int_{\varepsilon < |z - z_0| < \varepsilon_0} K_\mu(z) \cdot \psi^2(|z - z_0|) dm(z) < \infty \quad \forall z_0 \in \overline{D}$$

and $I_{z_0}(\varepsilon) \rightarrow \infty$ as $\varepsilon \rightarrow 0$. In other words, for the solvability of the Dirichlet problem (1.13) for the Beltrami equation (1.1) for all continuous $\varphi(\zeta) \neq \text{const}$, it is sufficient that the integral in (8.3) converges in the sense of the principal value for some nonnegative function $\psi(t)$ that is locally integrable over $(0, \varepsilon_0]$ but has a nonintegrable singularity at 0. The functions $\log^\lambda(e/|z - z_0|)$, $\lambda \in (0, 1)$, $z \in \mathbb{D}$, $z_0 \in \overline{\mathbb{D}}$, and $\psi(t) = 1/(t \log(e/t))$, $t \in (0, 1)$, show that the condition (8.3) is compatible with the condition $I_{z_0}(\varepsilon) \rightarrow \infty$ as $\varepsilon \rightarrow 0$. Furthermore, the condition (8.1) shows that it is sufficient for the solvability of the Dirichlet problem even that the integral in (8.3) was divergent in a controlled way.

PROOF OF LEMMA 8.1. Let F be a regular homeomorphic solution of the Beltrami equation (1.1) of the class $W_{\text{loc}}^{1,1}$ that exists by Lemma 4.1 in [45]. Note that $\overline{\mathbb{C}} \setminus D^*$, where $D^* = F(D)$, cannot consist of the single point ∞ because in the contrary case ∂D^* would be weakly flat. But then by Lemma 5.4 F should have a homeomorphic extension to \overline{D} that is impossible because ∂D is not a singleton. Moreover, the domain D^* is simply connected, see, e.g., either Lemma 5.3 in [17] or Lemma 6.5 in [29]. Thus, by the Riemann theorem, see, e.g., Theorem II.2.1 in [10], D^* can be mapped by a conformal mapping R onto the unit disk \mathbb{D} . The mapping $g = R \circ F$ is also a regular homeomorphic solution of the Beltrami equation of the class $W_{\text{loc}}^{1,1}$ that maps D onto \mathbb{D} . Furthermore, by Lemma 7.1 g admits a homeomorphic extension $g_* : \overline{D} \rightarrow \overline{\mathbb{D}}$ because \mathbb{D} has a weakly flat boundary and the Jordan domain D is locally connected on its boundary.

Let us find a solution of the Dirichlet problem (1.13) in the form $f = h \circ g$ where h is an analytic function in \mathbb{D} with the boundary condition

$$\lim_{z \rightarrow \zeta} \text{Re } h(z) = \varphi(g_*^{-1}(\zeta)) \quad \forall \zeta \in \partial \mathbb{D}.$$

By the Schwarz formula (see, e.g., Section 8, Chapter III, Part 3 in [15]), the analytic function h with $\text{Im } h(0) = 0$ can be calculated in \mathbb{D} through its real part on the boundary:

$$(8.4) \quad h(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \text{Re } \varphi \circ g_*^{-1}(\zeta) \cdot \frac{\zeta + z}{\zeta - z} \cdot \frac{d\zeta}{\zeta}.$$

We see that the function $f = h \circ g$ is the desired regular solution of the Dirichlet problem (1.13) for the Beltrami equation (1.1). \square

Choosing in Lemma 8.1 $\psi(t) = 1/(t \log(1/t))$, we obtain by Lemma 2.6 the following result.

THEOREM 8.3. *Let D be a Jordan domain and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L_{\text{loc}}^1(D)$. Suppose that $K_\mu^T(z, z_0) \leq Q_{z_0}(z)$ a.e. in $D \cap U_{z_0}$ for every point $z_0 \in \overline{D}$, a neighborhood U_{z_0} of z_0 and a function $Q_{z_0} : \mathbb{C} \rightarrow [0, \infty]$ in the class $\text{FMO}(z_0)$. Then the Beltrami equation (1.1)*

has a regular solution of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$.

REMARK 8.4. In particular, the conditions and the conclusion of Theorem 8.3 hold if either $Q_{z_0} \in \text{BMO}_{\text{loc}}$ or $Q_{z_0} \in W_{\text{loc}}^{1,2}$ because $W_{\text{loc}}^{1,2} \subset \text{VMO}_{\text{loc}}$, see, e.g., [3].

Since $K_\mu^T(z, z_0) \leq K_\mu(z)$ for all $z_0 \in \mathbb{C}$ and $z \in D$, we obtain the following consequence of Theorem 8.3.

COROLLARY 8.5. Let D be a Jordan domain and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and such that $K_\mu(z) \leq Q(z)$ a.e. in D for a function $Q : \mathbb{C} \rightarrow [0, \infty]$ in $\text{FMO}(\overline{D})$. Then the Beltrami equation (1.1) has a regular solution of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$.

COROLLARY 8.6. In particular, the conclusion of Theorem 8.3 holds if every point $z_0 \in \overline{D}$ is the Lebesgue point of a function $Q_{z_0} : \mathbb{C} \rightarrow [0, \infty]$ which is integrable in a neighborhood U_{z_0} of the point z_0 such that $K_\mu^T(z, z_0) \leq Q_{z_0}(z)$ a.e. in $D \cap U_{z_0}$.

We assume that $K_\mu^T(z, z_0)$ is extended by zero outside of D in the following consequences of Theorem 8.3.

COROLLARY 8.7. Let D be a Jordan domain and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L_{\text{loc}}^1(D)$ and

$$(8.5) \quad \lim_{\varepsilon \rightarrow 0} \int_{B(z_0, \varepsilon)} K_\mu^T(z, z_0) dm(z) < \infty \quad \forall z_0 \in \overline{D}.$$

Then the Beltrami equation (1.1) has a regular solution of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$.

Similarly, choosing in Lemma 8.1 the function $\psi(t) = 1/t$, we come to the following statement.

THEOREM 8.8. Let D be a Jordan domain and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L_{\text{loc}}^1(D)$. Suppose that

$$(8.6) \quad \int_{\varepsilon < |z - z_0| < \varepsilon_0} K_\mu^T(z, z_0) \frac{dm(z)}{|z - z_0|^2} = o\left(\left[\log \frac{1}{\varepsilon}\right]^2\right) \quad \forall z_0 \in \overline{D}$$

as $\varepsilon \rightarrow 0$ for some $\varepsilon_0 = \delta(z_0) \in (0, d(z_0))$ where $d(z_0) = \sup_{z \in D} |z - z_0|$. Then the Beltrami equation (1.1) has a regular solution of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$.

REMARK 8.9. Choosing in Lemma 8.1 the function $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$, we are able to replace (8.6) by

$$(8.7) \quad \int_{\varepsilon < |z - z_0| < \varepsilon_0} \frac{K_\mu^T(z, z_0) dm(z)}{\left(|z - z_0| \log \frac{1}{|z - z_0|}\right)^2} = o\left(\left[\log \log \frac{1}{\varepsilon}\right]^2\right)$$

In general, we are able to give here the whole scale of the corresponding conditions in log using functions $\psi(t)$ of the form $1/(t \log 1/t \cdot \log \log 1/t \cdot \dots \cdot \log \dots \log 1/t)$.

Arguing similarly to the proof of Lemma 8.1, we obtain on the basis of Theorem 7.9 the following result.

THEOREM 8.10. *Let D be a Jordan domain in \mathbb{C} and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L^1_{\text{loc}}(D)$. Suppose that*

$$(8.8) \quad \int_0^{\delta(z_0)} \frac{dr}{\|K_\mu^T\|_1(z_0, r)} = \infty \quad \forall z_0 \in \overline{D}$$

for some $\delta(z_0) \in (0, d(z_0))$ where $d(z_0) = \sup_{z \in D} |z - z_0|$ and

$$(8.9) \quad \|K_\mu^T\|_1(z_0, r) = \int_{D \cap S(z_0, r)} K_\mu^T(z, z_0) |dz|.$$

Then the Beltrami equation (1.1) has a regular solution f of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$.

COROLLARY 8.11. *Let D be a Jordan domain and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L^1_{\text{loc}}(D)$. Suppose that*

$$(8.10) \quad k_{z_0}(\varepsilon) = O\left(\log \frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \rightarrow 0 \quad \forall z_0 \in \overline{D}$$

where $k_{z_0}(\varepsilon)$ is the average of the function $K_\mu^T(z, z_0)$ over $S(z_0, \varepsilon)$. Then the Beltrami equation (1.1) has a regular solution of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$.

REMARK 8.12. In particular, the conclusion of Corollary 8.11 holds if

$$(8.11) \quad K_\mu^T(z, z_0) = O\left(\log \frac{1}{|z - z_0|}\right) \quad \text{as } z \rightarrow z_0 \quad \forall z_0 \in \overline{D}.$$

COROLLARY 8.13. *Let D be a Jordan domain and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L^1_{\text{loc}}(D)$ and*

$$(8.12) \quad k_{z_0}(\varepsilon) = O\left(\left[\log \frac{1}{\varepsilon} \cdot \log \log \frac{1}{\varepsilon} \cdot \dots \cdot \log \dots \log \frac{1}{\varepsilon}\right]\right) \quad \forall z_0 \in \overline{D}$$

as $\varepsilon \rightarrow 0$, where $k_{z_0}(\varepsilon)$ is the average of the function $K_\mu^T(z, z_0)$ over $S(z_0, \varepsilon)$. Then the Beltrami equation (1.1) has a regular solution of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$.

Finally, combining Theorems 2.7 and 8.10, we obtain the following.

THEOREM 8.14. *Let D be a Jordan domain and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L^1_{\text{loc}}(D)$. Suppose that*

$$(8.13) \quad \int_{D \cap U_{z_0}} \Phi_{z_0}(K_\mu^T(z, z_0)) \, dm(z) < \infty \quad \forall z_0 \in \overline{D}$$

for a neighborhood U_{z_0} of z_0 and a convex non-decreasing function $\Phi_{z_0} : [0, \infty] \rightarrow [0, \infty]$ such that

$$(8.14) \quad \int_{\delta(z_0)}^{\infty} \frac{d\tau}{\tau \Phi_{z_0}^{-1}(\tau)} = \infty$$

for some $\delta(z_0) > \Phi_{z_0}(0)$. Then the Beltrami equation (1.1) has a regular solution of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$.

COROLLARY 8.15. *In particular, the conclusion of Theorem 8.14 holds if*

$$(8.15) \quad \int_{D \cap U_{z_0}} e^{\alpha(z_0) K_{\mu}^T(z, z_0)} dm(z) < \infty \quad \forall z_0 \in \overline{D}$$

for some $\alpha(z_0) > 0$ and a neighborhood U_{z_0} of the point z_0 .

Since $K_{\mu}^T(z, z_0) \leq K_{\mu}(z)$ for all $z_0 \in \mathbb{C}$ and $z \in D$, we obtain the following consequence of Theorem 8.14.

COROLLARY 8.16. *Let D be a Jordan domain and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that*

$$(8.16) \quad \int_D \Phi(K_{\mu}(z)) dm(z) < \infty$$

for a convex non-decreasing function $\Phi : [0, \infty] \rightarrow [0, \infty]$. If

$$(8.17) \quad \int_{\delta}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty$$

for some $\delta > \Phi(0)$. Then the Beltrami equation (1.1) has a regular solution of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$.

REMARK 8.17. By the Stoilow theorem, see, e.g., [51], a regular solution f of the Dirichlet problem (1.13) for the Beltrami equation (1.1) with $K_{\mu} \in L_{\text{loc}}^1(D)$ can be represented in the form $f = h \circ F$ where h is an analytic function and F is a homeomorphic regular solution of (1.1) in the class $W_{\text{loc}}^{1,1}$. Thus, by Theorem 5.1 in [47] the condition (8.17) is not only sufficient but also necessary to have a regular solution of the Dirichlet problem (1.13) for an arbitrary Beltrami equation (1.1) with the integral constraints (8.16) for any non-constant continuous function $\varphi : \partial D \rightarrow \mathbb{R}$, see also Remark 2.8.

9. On pseudoregular solutions in multiply connected domains

As it was first noted by Bojarski, see, e.g., section 6 of Chapter 4 in [53], that in the case of multiply connected domains the Dirichlet problem for the Beltrami equation, generally speaking, has no solutions in the class of continuous (simply-valued) functions. Hence it is arose the question: whether the existence of solutions for the Dirichlet problem can be obtained in a wider function class for the case? It is turned out to be that this is possible in the class of functions having a certain number of poles at prescribed points in D . More precisely, for $\varphi(\zeta) \not\equiv \text{const}$, a **pseudoregular solution** of the problem is a continuous (in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$) discrete open mapping $f : D \rightarrow \overline{\mathbb{C}}$ in the class $W_{\text{loc}}^{1,1}$ (outside of these poles) with the Jacobian $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 \neq 0$ a.e. satisfying (1.1) a.e. and condition (1.13).

As above, we assume in the following lemma that $K_\mu^T(z, z_0)$ is extended by zero outside of the domain D .

LEMMA 9.1. *Let D be a bounded domain in \mathbb{C} whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves. Suppose that $\mu : D \rightarrow \mathbb{C}$ is a measurable function with $|\mu(z)| < 1$ a.e., $K_\mu(z) \in L_{\text{loc}}^1(D)$, $K_\mu^T(z, z_0)$ is integrable over $D \cap S(z_0, r)$ for a.e. $r \in (0, \varepsilon_0)$ and*

$$(9.1) \quad \int_{\varepsilon < |z - z_0| < \varepsilon_0} K_\mu^T(z, z_0) \cdot \psi_{z_0, \varepsilon}^2(|z - z_0|) dm(z) = o(I_{z_0}^2(\varepsilon)) \quad \text{as } \varepsilon \rightarrow 0 \quad \forall z_0 \in \overline{D}$$

for some $\varepsilon_0 \in (0, \delta_0)$ where $\delta_0 = \delta(z_0) = \sup_{z \in D} |z - z_0|$ and $\psi_{z_0, \varepsilon}(t)$ is a family of non-negative measurable (by Lebesgue) functions on $(0, \infty)$ such that

$$(9.2) \quad I_{z_0}(\varepsilon) := \int_{\varepsilon}^{\varepsilon_0} \psi_{z_0, \varepsilon}(t) dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Then the Beltrami equation (1.1) has a pseudoregular solution of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$, $\varphi(\zeta) \not\equiv \text{const}$, with poles at n prescribed points $z_i \in D$, $i = 1, \dots, n$.

PROOF. Let F be a regular homeomorphic solution of the Beltrami equation (1.1) of the class $W_{\text{loc}}^{1,1}$ that exists by Lemma 4.1 in [45]. Consider $D^* = f(D)$. Note that ∂D^* has n connected components Γ_i , $i = 1, \dots, n$ that correspond in the natural way to connected components of ∂D , the Jordan curves γ_i , see, e.g., either Lemma 5.3 in [17] or Lemma 6.5 in [29].

Thus, by Theorem V.6.2 in [10] the domain D_* can be mapped by a conformal map R onto a circular domain \mathbb{D}_* whose boundary consists of n circles or points, i.e. \mathbb{D}_* has a weakly flat boundary. Note that the mapping $g := R \circ F$ is a regular homeomorphic solution of the Beltrami equation in the class $W_{\text{loc}}^{1,1}$ admitting a homeomorphic extension $g_* : \overline{D} \rightarrow \overline{\mathbb{D}_*}$ by Lemma 7.1.

Let us find a solution of the Dirichlet problem (1.13) in the form $f = h \circ g$ where h is a meromorphic function with n poles at the prescribed points $w_i = g(z_i)$, $i = 1, \dots, n$ in \mathbb{D}_* with the boundary condition

$$\lim_{w \rightarrow \zeta} \text{Re } h(w) = \varphi(g_*^{-1}(\zeta)) \quad \forall \zeta \in \partial \mathbb{D}_*.$$

Such a function h exists by Theorem 4.14 in [53].

We see that the function $f = h \circ g$ is the desired pseudoregular solution of the Dirichlet problem (1.13) for the Beltrami equation (1.1) with n poles just at these prescribed points z_i , $i = 1, \dots, n$. \square

Arguing similarly to the last section, by the special choice of the functional parameter ψ in Lemma 9.1, we obtain the following results.

THEOREM 9.2. *Let D be a bounded domain in \mathbb{C} whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L^1_{\text{loc}}(D)$. Suppose that, for every point $z_0 \in \overline{D}$ and a neighborhood U_{z_0} of z_0 , $K_\mu^T(z, z_0) \leq Q_{z_0}(z)$ a.e. in $D \cap U_{z_0}$ for a function $Q_{z_0} : \mathbb{C} \rightarrow [0, \infty]$ in the class $\text{FMO}(z_0)$. Then the Beltrami equation (1.1) has a pseudoregular solution of the Dirichlet problem (1.13) for every continuous function $\varphi : \partial D \rightarrow \mathbb{R}$, $\varphi(\zeta) \not\equiv \text{const}$, with poles at n prescribed points in D .*

REMARK 9.3. In particular, the conditions and the conclusion of Theorem 9.2 hold if either $Q_{z_0} \in \text{BMO}_{\text{loc}}$ or $Q_{z_0} \in W^{1,2}_{\text{loc}}$ because $W^{1,2}_{\text{loc}} \subset \text{VMO}_{\text{loc}}$.

COROLLARY 9.4. *Let D be a bounded domain in \mathbb{C} whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu(z) \leq Q(z)$ a.e. in \overline{D} for a function $Q : \mathbb{C} \rightarrow [0, \infty]$ in the class $\text{FMO}(\overline{D})$. Then the Beltrami equation (1.1) has a pseudoregular solution of the Dirichlet problem (1.13) for every continuous function $\varphi : \partial D \rightarrow \mathbb{R}$, $\varphi(\zeta) \not\equiv \text{const}$, with poles at n prescribed points in D .*

COROLLARY 9.5. *In particular, the conclusion of Theorem 9.2 holds if every point $z_0 \in \overline{D}$ is the Lebesgue point of a function $Q_{z_0} : \mathbb{C} \rightarrow [0, \infty]$ which is integrable in a neighborhood U_{z_0} of the point z_0 such that $K_\mu^T(z, z_0) \leq Q_{z_0}(z)$ a.e. in $D \cap U_{z_0}$.*

COROLLARY 9.6. *In particular, the conclusion of Theorem 9.2 holds if*

$$(9.3) \quad \overline{\lim}_{\varepsilon \rightarrow 0} \int_{B(z_0, \varepsilon)} K_\mu^T(z, z_0) dm(z) < \infty \quad \forall z_0 \in \overline{D}.$$

As above, here we assume that $K_\mu^T(z, z_0)$ is extended by zero outside of D .

THEOREM 9.7. *Let D be a bounded domain in \mathbb{C} whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L^1_{\text{loc}}(D)$. Suppose that*

$$(9.4) \quad \int_{\varepsilon < |z - z_0| < \varepsilon_0} K_\mu^T(z, z_0) \frac{dm(z)}{|z - z_0|^2} = o\left(\left[\log \frac{1}{\varepsilon}\right]^2\right) \quad \forall z_0 \in \overline{D}$$

as $\varepsilon \rightarrow 0$ for some $\varepsilon_0 = \delta(z_0) \in (0, d(z_0))$ where $d(z_0) = \sup_{z \in D} |z - z_0|$. Then the Beltrami equation (1.1) has a pseudoregular solution of the Dirichlet problem (1.13) for every continuous function $\varphi : \partial D \rightarrow \mathbb{R}$, $\varphi(\zeta) \not\equiv \text{const}$, with poles at n prescribed points in D .

REMARK 9.8. Choosing in Lemma 9.1 the function $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$, we are able to replace (9.4) by

$$(9.5) \quad \int_{\varepsilon < |z - z_0| < \varepsilon_0} \frac{K_\mu^T(z, z_0) dm(z)}{\left(|z - z_0| \log \frac{1}{|z - z_0|}\right)^2} = o\left(\left[\log \log \frac{1}{\varepsilon}\right]^2\right)$$

In general, we are able to give here the whole scale of the corresponding conditions in log using functions $\psi(t)$ of the form $1/(t \log 1/t \cdot \log \log 1/t \cdot \dots \cdot \log \dots \log 1/t)$.

Arguing similarly to the proof of Lemma 9.1, we obtain on the basis of the Theorem 7.9 the following.

THEOREM 9.9. *Let D be a bounded domain in \mathbb{C} whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L_{\text{loc}}^1(D)$. Suppose that*

$$(9.6) \quad \int_0^{\delta(z_0)} \frac{dr}{\|K_\mu^T\|_1(z_0, r)} = \infty \quad \forall z_0 \in \overline{D}$$

for some $\delta(z_0) \in (0, d(z_0))$ where $d(z_0) = \sup_{z \in D} |z - z_0|$ and

$$(9.7) \quad \|K_\mu^T\|_1(z_0, r) = \int_{D \cap S(z_0, r)} K_\mu^T(z, z_0) |dz|.$$

Then the Beltrami equation (1.1) has a pseudoregular solution of the Dirichlet problem (1.13) for every continuous function $\varphi : \partial D \rightarrow \mathbb{R}$, $\varphi(\zeta) \not\equiv \text{const}$, with poles at n prescribed points in D .

COROLLARY 9.10. *Let D be a bounded domain in \mathbb{C} whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L_{\text{loc}}^1(D)$. Suppose that*

$$(9.8) \quad k_{z_0}(\varepsilon) = O\left(\log \frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \rightarrow 0 \quad \forall z_0 \in \overline{D},$$

where $k_{z_0}(\varepsilon)$ is the average of the function $K_\mu^T(z, z_0)$ over the circle $\{z \in \mathbb{C} : |z - z_0| = \varepsilon\}$. Then the Beltrami equation (1.1) has a pseudoregular solution of the Dirichlet problem (1.13) for every continuous function $\varphi : \partial D \rightarrow \mathbb{R}$, $\varphi(\zeta) \not\equiv \text{const}$, with poles at n prescribed points in D .

REMARK 9.11. In particular, the conclusion of Corollary 9.10 holds if

$$(9.9) \quad K_\mu^T(z, z_0) = O\left(\log \frac{1}{|z - z_0|}\right) \quad \text{as } z \rightarrow z_0 \quad \forall z_0 \in \overline{D}.$$

COROLLARY 9.12. *Let D be a bounded domain in \mathbb{C} whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu(z) \in L_{\text{loc}}^1(D)$. Suppose that*

$$(9.10) \quad k_{z_0}(\varepsilon) = O\left(\left[\log \frac{1}{\varepsilon} \cdot \log \log \frac{1}{\varepsilon} \cdot \dots \cdot \log \dots \log \frac{1}{\varepsilon}\right]\right) \quad \forall z_0 \in \overline{D}$$

where $k_{z_0}(\varepsilon)$ is the average of the function $K_\mu^T(z, z_0)$ over the circle $\{z \in \mathbb{C} : |z - z_0| = \varepsilon\}$. Then the Beltrami equation (1.1) has a pseudoregular solution of the Dirichlet problem (1.13) for every continuous function $\varphi : \partial D \rightarrow \mathbb{R}$, $\varphi(\zeta) \not\equiv \text{const}$, with poles at n prescribed points in D .

THEOREM 9.13. *Let D be a bounded domain in \mathbb{C} whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves and let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L_{\text{loc}}^1(D)$. Suppose that*

$$(9.11) \quad \int_{D \cap U_{z_0}} \Phi_{z_0}(K_\mu^T(z, z_0)) \, dm(z) < \infty \quad \forall z_0 \in \overline{D}$$

for a neighborhood U_{z_0} of the point z_0 and a convex non-decreasing function $\Phi_{z_0} : [0, \infty] \rightarrow [0, \infty]$ such that

$$(9.12) \quad \int_{\delta(z_0)}^{\infty} \frac{d\tau}{\tau \Phi_{z_0}^{-1}(\tau)} = \infty$$

for some $\delta(z_0) > \Phi_{z_0}(0)$. Then the Beltrami equation (1.1) has a pseudoregular solution of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$, $\varphi(\zeta) \not\equiv \text{const}$, with poles at n prescribed inner points of D .

COROLLARY 9.14. *In particular, the conclusion of Theorem 9.13 holds if*

$$(9.13) \quad \int_{D \cap U_{z_0}} e^{\alpha(z_0) K_\mu^T(z, z_0)} \, dm(z) < \infty \quad \forall z_0 \in \overline{D}$$

for some $\alpha(z_0) > 0$ and a neighborhood U_{z_0} of the point z_0 .

COROLLARY 9.15. *Let D be a bounded domain in \mathbb{C} whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves and let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that*

$$(9.14) \quad \int_D \Phi(K_\mu(z)) \, dm(z) < \infty$$

for a convex non-decreasing function $\Phi : [0, \infty] \rightarrow [0, \infty]$. If

$$(9.15) \quad \int_{\delta}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty$$

for some $\delta > \Phi(0)$. Then the Beltrami equation (1.1) has a pseudoregular solution of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$, $\varphi(\zeta) \not\equiv \text{const}$, with poles at n prescribed inner points in D .

10. On multi-valued solutions in finitely connected domains

In finitely connected domains D in \mathbb{C} , in addition to pseudoregular solutions, the Dirichlet problem (1.13) for the Beltrami equation (1.1) admits multi-valued solutions in the spirit of the theory of multi-valued analytic functions. We say that a discrete open mapping $f : B(z_0, \varepsilon_0) \rightarrow \mathbb{C}$, where $B(z_0, \varepsilon_0) \subseteq D$, is a **local regular solution of the equation** (1.1) if $f \in W_{\text{loc}}^{1,1}$, $J_f(z) \neq 0$ and f satisfies (1.1) a.e. in $B(z_0, \varepsilon_0)$.

The local regular solutions $f : B(z_0, \varepsilon_0) \rightarrow \mathbb{C}$ and $f_* : B(z_*, \varepsilon_*) \rightarrow \mathbb{C}$ of the equation (1.1) will be called extension of each to other if there is a finite chain of such solutions $f_i : B(z_i, \varepsilon_i) \rightarrow \mathbb{C}$, $i = 1, \dots, m$, that $f_1 = f_0$, $f_m = f_*$ and $f_i(z) \equiv f_{i+1}(z)$ for $z \in E_i := B(z_i, \varepsilon_i) \cap B(z_{i+1}, \varepsilon_{i+1}) \neq \emptyset$, $i = 1, \dots, m-1$. A collection of local regular solutions $f_j : B(z_j, \varepsilon_j) \rightarrow \mathbb{C}$, $j \in J$, will be called a **multi-valued solution** of the equation (1.1) in D if the disks $B(z_j, \varepsilon_j)$ cover the whole domain D and f_j are extensions of each to other through the collection and the collection is maximal by inclusion. A multi-valued solution of the equation (1.1) will be called a **multi-valued solution of the Dirichlet problem** (1.13) if $u(z) = \operatorname{Re} f(z) = \operatorname{Re} f_j(z)$, $z \in B(z_j, \varepsilon_j)$, $j \in J$, is a simply-valued function in D satisfying the condition $\lim_{z \in \zeta} u(z) = \varphi(\zeta)$ for all $\zeta \rightarrow \partial D$.

As usual, we assume in the following lemma that $K_\mu^T(\cdot, z_0)$ is extended by zero outside of the domain D .

LEMMA 10.1. *Let D be a bounded domain in \mathbb{C} whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves. Suppose that $\mu : D \rightarrow \mathbb{C}$ is a measurable function with $|\mu(z)| < 1$ a.e., $K_\mu \in L_{\text{loc}}^1(D)$, $K_\mu^T(z, z_0)$ is integrable over $D \cap S(z_0, r)$ for a.e. $r \in (0, \varepsilon_0)$ and*

$$(10.1) \quad \int_{\varepsilon < |z - z_0| < \varepsilon_0} K_\mu^T(z, z_0) \cdot \psi_{z_0, \varepsilon}^2(|z - z_0|) dm(z) = o(I_{z_0}^2(\varepsilon)) \quad \text{as } \varepsilon \rightarrow 0 \quad \forall z_0 \in \overline{D}$$

for some $\varepsilon_0 \in (0, \delta_0)$ where $\delta_0 = \delta(z_0) = \sup_{z \in D} |z - z_0|$ and $\psi_{z_0, \varepsilon}(t)$ is a family of non-negative measurable (by Lebesgue) functions on $(0, \infty)$ such that

$$(10.2) \quad I_{z_0}(\varepsilon) := \int_{\varepsilon}^{\varepsilon_0} \psi_{z_0, \varepsilon}(t) dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Then the Beltrami equation (1.1) has a multi-valued solutions of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$.

PROOF. Let F be a regular homeomorphic solution of the Beltrami equation (1.1) in the class $W_{\text{loc}}^{1,1}$ that exists by Lemma 4.1 in [45]. As it was showed under the proof of Lemma 9.1, we may assume that $D_* := F(D)$ is a circular domain and that F can be extended to a homeomorphism $F_* : \overline{D} \rightarrow \overline{D}_*$. Let $u : D_* \rightarrow \mathbb{R}$ be a harmonic function such that

$$\lim_{w \rightarrow \zeta} u(w) = \varphi(F_*^{-1}(\zeta)) \quad \forall \zeta \in \partial D^*$$

whose existence is well-known, see, e.g., Section 3 of Chapter VI in [10].

Let $z_0 \in D$, $B_0 := B(z_0, \varepsilon_0) \subseteq D$ for some $\varepsilon_0 > 0$. Then the domain $D_0 = F(B_0)$ is simply connected and hence there is a harmonic function $v(w)$ such that $h(w) = u(w) + iv(w)$ is a holomorphic function which is unique up to an additive constant, see, e.g., Theorem 1 in Section 7 of Chapter III, Part 3 in [15]. Note that $f_0 := h \circ F|_{B_0}$ is a local regular solution of the Beltrami equation (1.1). Note that the function h can be extended to a multi-valued analytic function H in the domain D_* and, thus, $H \circ F$ gives the desired multi-valued solutions of the Dirichlet problem (1.13) for the Beltrami equation (1.1). \square

In particular, by Lemma 10.1 above and Lemma 2.6 we obtain the following.

THEOREM 10.2. *Let D be a bounded domain in \mathbb{C} whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L^1_{\text{loc}}(D)$. Suppose that, for every point $z_0 \in \overline{D}$ and a neighborhood U_{z_0} of z_0 , $K_\mu^T(z, z_0) \leq Q_{z_0}(z)$ a.e. in $D \cap U_{z_0}$ for a function $Q_{z_0} : \mathbb{C} \rightarrow [0, \infty]$ in the class $\text{FMO}(z_0)$. Then the Beltrami equation (1.1) has a multi-valued solutions of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$.*

REMARK 10.3. In particular, the conditions and the conclusion of Theorem 10.2 hold if either $Q_{z_0} \in \text{BMO}_{\text{loc}}$ or $Q_{z_0} \in W^{1,2}_{\text{loc}}$ because $W^{1,2}_{\text{loc}} \subset \text{VMO}_{\text{loc}}$.

COROLLARY 10.4. *Let D be a bounded domain in \mathbb{C} whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and such that $K_\mu(z) \leq Q(z)$ a.e. in D for a function $Q : \mathbb{C} \rightarrow [0, \infty]$ in $\text{FMO}(\overline{D})$. Then the Beltrami equation (1.1) has a multi-valued solutions of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$.*

COROLLARY 10.5. *In particular, the conclusion of Theorem 10.2 holds if every point $z_0 \in \overline{D}$ is Lebesgue point of a function $Q_{z_0} : \mathbb{C} \rightarrow [0, \infty]$ which is integrable in a neighborhood U_{z_0} of the point z_0 such that $K_\mu^T(z, z_0) \leq Q_{z_0}(z)$ a.e. in $D \cap U_{z_0}$.*

We assume that $K_\mu^T(z, z_0)$ is extended by zero outside of D in the next consequences of Theorem 10.2.

COROLLARY 10.6. *Let D be a bounded domain in \mathbb{C} whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L^1_{\text{loc}}(D)$. Suppose that*

$$(10.3) \quad \overline{\lim}_{\varepsilon \rightarrow 0} \int_{B(z_0, \varepsilon)} K_\mu^T(z, z_0) dm(z) < \infty \quad \forall z_0 \in \overline{D}.$$

Then the Beltrami equation (1.1) has a multi-valued solutions of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$.

THEOREM 10.7. *Let D be a bounded domain in \mathbb{C} whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L^1_{\text{loc}}(D)$. Suppose that*

$$(10.4) \quad \int_{\varepsilon < |z - z_0| < \varepsilon_0} K_\mu^T(z, z_0) \frac{dm(z)}{|z - z_0|^2} = o\left(\left[\log \frac{1}{\varepsilon}\right]^2\right) \quad \forall z_0 \in \overline{D}$$

as $\varepsilon \rightarrow 0$ for some $\varepsilon_0 = \delta(z_0) \in (0, d(z_0))$ where $d(z_0) = \sup_{z \in D} |z - z_0|$. Then the Beltrami equation (1.1) has a pseudoregular solution of the Dirichlet problem (1.13) for every continuous function $\varphi : \partial D \rightarrow \mathbb{R}$, $\varphi(\zeta) \not\equiv \text{const}$, with poles at n prescribed points in D .

REMARK 10.8. Choosing in Lemma 10.1 the function $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$, we are able to replace (10.4) by

$$(10.5) \quad \int_{\varepsilon < |z - z_0| < \varepsilon_0} \frac{K_\mu^T(z, z_0) dm(z)}{\left(|z - z_0| \log \frac{1}{|z - z_0|}\right)^2} = o\left(\left[\log \log \frac{1}{\varepsilon}\right]^2\right)$$

In general, we are able to give here the whole scale of the corresponding conditions in log using functions $\psi(t)$ of the form $1/(t \log 1/t \cdot \log \log 1/t \cdot \dots \cdot \log \dots \log 1/t)$.

Arguing similarly to the proof of Lemma 10.1, we obtain on the basis of Theorem 7.9 the next result.

THEOREM 10.9. *Let D be a bounded domain in \mathbb{C} whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L^1_{\text{loc}}(D)$. Suppose that*

$$(10.6) \quad \int_0^{\delta(z_0)} \frac{dr}{\|K_\mu^T\|_1(z_0, r)} = \infty \quad \forall z_0 \in \overline{D}$$

for some $\delta(z_0) \in (0, d(z_0))$ where $d(z_0) = \sup_{z \in D} |z - z_0|$ and

$$(10.7) \quad \|K_\mu^T\|_1(z_0, r) = \int_{D \cap S(z_0, r)} K_\mu^T(z, z_0) |dz|.$$

Then the Beltrami equation (1.1) has a multi-valued solutions of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$.

COROLLARY 10.10. *Let D be a bounded domain in \mathbb{C} whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L^1_{\text{loc}}(D)$. Suppose that*

$$(10.8) \quad k_{z_0}(\varepsilon) = O\left(\log \frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \rightarrow 0 \quad \forall z_0 \in \overline{D}$$

where $k_{z_0}(\varepsilon)$ is the average of the function $K_\mu^T(z, z_0)$ over $S(z_0, \varepsilon)$. Then the Beltrami equation (1.1) has a multi-valued solutions of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$.

REMARK 10.11. In particular, the conclusion of Corollary 10.10 holds if

$$(10.9) \quad K_\mu^T(z, z_0) = O\left(\log \frac{1}{|z - z_0|}\right) \quad \text{as } z \rightarrow z_0 \quad \forall z_0 \in \overline{D}.$$

COROLLARY 10.12. Let D be a bounded domain in \mathbb{C} whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L_{\text{loc}}^1(D)$. Suppose that

$$(10.10) \quad k_{z_0}(\varepsilon) = O\left(\left[\log \frac{1}{\varepsilon} \cdot \log \log \frac{1}{\varepsilon} \cdot \dots \cdot \log \dots \log \frac{1}{\varepsilon}\right]\right) \quad \forall z_0 \in \overline{D}$$

where $k_{z_0}(\varepsilon)$ is the average of the function $K_\mu^T(z, z_0)$ over $S(z_0, \varepsilon)$. Then the Beltrami equation (1.1) has a multi-valued solutions of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$.

THEOREM 10.13. Let D be a bounded domain in \mathbb{C} whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves and $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that $K_\mu \in L_{\text{loc}}^1(D)$. Suppose that

$$(10.11) \quad \int_D \Phi_{z_0}(K_\mu^T(z, z_0)) \, dm(z) < \infty \quad \forall z_0 \in \overline{D}$$

for a convex non-decreasing function $\Phi_{z_0} : [0, \infty] \rightarrow [0, \infty]$ such that

$$(10.12) \quad \int_{\delta(z_0)}^\infty \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty$$

for some $\delta(z_0) > \Phi_{z_0}(0)$. Then the Beltrami equation (1.1) has a multi-valued solutions of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$.

COROLLARY 10.14. In particular, the conclusion of Theorem 10.13 holds if

$$(10.13) \quad \int_{D \cap U_{z_0}} e^{\alpha(z_0) K_\mu^T(z, z_0)} \, dm(z) < \infty \quad \forall z_0 \in \overline{D}$$

for some $\alpha(z_0) > 0$ and a neighborhood U_{z_0} of the point z_0 .

COROLLARY 10.15. Let D be a bounded domain in \mathbb{C} whose boundary consists of $n \geq 2$ mutually disjoint Jordan curves and $\mu : D \rightarrow \mathbb{C}$ be a measurable function such that $|\mu(z)| < 1$ a.e. and

$$(10.14) \quad \int_D \Phi(K_\mu(z)) \, dm(z) < \infty$$

for a convex non-decreasing function $\Phi : [0, \infty] \rightarrow [0, \infty]$. If

$$(10.15) \quad \int_\delta^\infty \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty$$

for some $\delta > \Phi(0)$. Then the Beltrami equation (1.1) has a multi-valued solutions of the Dirichlet problem (1.13) for each continuous function $\varphi : \partial D \rightarrow \mathbb{R}$.

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